INTERIOR CONTINUITY OF TWO-DIMENSIONAL WEAKLY STATIONARY-HARMONIC MULTIPLE-VALUED FUNCTIONS

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Abstract. In his big regularity paper, Almgren has proven the regularity theorem for mass-minimizing integral currents. One key step in his paper is to derive the regularity of Dirichlet-minimizing $\mathbf{Q}_Q(\mathbb{R}^n)$-valued functions in the Sobolev space $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$, where the domain Ω is open in $\mathbb{R}^m$. In this article, we introduce the class of weakly stationary-harmonic $\mathbf{Q}_Q(\mathbb{R}^n)$-valued functions. These functions are the critical points of Dirichlet integral under smooth domain-variations and range-variations. We prove that if Ω is a two-dimensional domain in $\mathbb{R}^2$ and $f \in \mathcal{Y}_2\left(\Omega, \mathbf{Q}_Q(\mathbb{R}^n)\right)$ is weakly stationary-harmonic, then f is continuous in the interior of the domain Ω.

1. Introduction

In his big regularity paper [1], Almgren proved the regularity theorem for mass-minimizing integral currents. More precisely, Almgren showed that any mass-minimizing integral current is smooth except on a singular subset of co-dimension two. One key step in [1] is to derive the regularity of Dirichlet-minimizing multiple-valued functions in the Sobolev space $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ of multiple-valued functions (see (2.6) for its definition). This is a key step because Almgren used this class of multiple-valued functions to approximate mass-minimizing integral currents, whose regularity hence inherits that of Dirichlet-minimizing multiple-valued functions. In [1], any Dirichlet-minimizing multiple-valued function, $f : \Omega \subset \mathbb{R}^m \to \mathbf{Q}_Q(\mathbb{R}^n)$, is shown to be Hölder continuous at any interior point of Ω and smooth outside a closed subset Σ, whose co-dimension is least two (in the sense of Hausdorff measure). Moreover, as $\Omega \subset \mathbb{R}^2$, the closed subset Σ is consisted of isolated points. Dirichlet-minimizing multiple-valued functions were further investigated in [18] by Zhu as $\Omega \subset \mathbb{R}^2$ and in [16] by Spadaro as $\Omega \subset \mathbb{C}^m$. The reader is also referred to a recent article by De Lellis and Spadaro in [4], which makes Almgren's theory of multiple-valued functions more accessible and provides some new points of view on the subject, e.g., intrinsic theory of the metric space $\mathbf{Q}_Q(\mathbb{R}^n)$.

Although the theory of multiple-valued functions was originally developed for the purpose of studying the regularity of mass-minimizing integral currents in [1], we

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found that the theory itself is interesting enough and deserves further investigation as an independent topic in calculus of variations. In [11], Mattila investigated a class of elliptic variational integrals of multiple-valued functions (interior Hölder continuity was derived in the 2-dimensional case). Along the direction of [11], De Lellis, Focardi and Spadaro in [3] further characterized the semicontinuity of certain elliptic integrals of multiple-valued functions. In [7], Goblet and Zhu studied the regularity of Dirichlet nearly minimizing multiple-valued functions. On the other hand, Almgren's multiple-valued functions have also been used to formulate certain (stable) branched minimal surfaces and minimal hypersurfaces by Rosales, Simon, Wickramasekera, et al (e.g., see [12], [13], [15], [17]). Their research projects provide an approach to investigate more details of the local behavior around singularities of minimal surfaces and minimal hypersurfaces, which are not area-minimizing but could be formulated as two-valued minimal graphs.

In this article, we propose studying a bigger class of multiple-valued functions, called `weakly stationary-harmonic` multiple-valued functions, which are the critical points of Dirichlet integral with respect to smooth domain-variations and range-variations in the Sobolev space $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$. The reader might wonder that one could treat this variational problem by a formulation of standard stationary harmonic maps into metric spaces. Indeed, this is what we thought at the beginning as we started this project. However, there are several difficulties when one tries to apply the methods in the (partial) regularity theory for stationary harmonic maps. The main difficulty comes from the fact that the metric space $\mathbf{Q}_Q(\mathbb{R}^n)$ could only be embedded into Euclidean spaces as a bi-Lipschitz (polyhedral cone) submanifold, which is not negatively curved and is lack of sufficient smoothness. Thus, we could not follow the typical definitions of stationary harmonic maps and the approach studying their (partial) regularity in literature here. We prove that if $f \in \mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ is weakly stationary-harmonic and $\Omega \subset \mathbb{R}^2$, then f is continuous in the interior of the domain Ω.

There are two parts in the proof of our main result in Theorem 1. In the first part, we apply the domain-variations of f to show that the Hopf differential of $\boldsymbol{\xi}_0 \circ f$ (denoted as $\Phi : \Omega \to \mathbb{C}$) is a holomorphic function, where $\boldsymbol{\xi}_0 : \mathbf{Q}_Q(\mathbb{R}^n) \to \mathbb{R}^{nQ}$ is any Lipschitz map defined in (2.4). This trick has been used in the 2-dimensional harmonic map theory with very general target spaces, e.g., non-positively curved metric spaces (cf. [14]). Then, by applying the other trick in Grüter's paper [9], there is an induced harmonic function $h : \Omega \to \mathbb{R}^2$ so that $(\boldsymbol{\xi}_0 \circ f, h) : \Omega \to \mathbb{R}^{nQ+2}$ is weakly conformal. This weakly conformal condition put us in a position to adapt the method in Grüter's paper [8] (studying the regularity of weak conformal H-surfaces) into the class of multiple-valued functions of this article in the second part.

In the second part, one key step is to establish a “global” monotonicity formula (see Step 2 of the proof of Lemma 3) from proper range-variations of f. This part contains the main difficulty of this article as we apply Grüter's approach in [8] to our case. Notice that, one could not simply perturb a multiple-valued function by adding a test function belonging to a proper class of functions (just like in PDE theory) or by further projections into a smooth submanifold (just like in the theory of stationary harmonic maps). This difficulty is due to the lack of structure for algebraic operations (e.g., addition and subtraction) in $\mathbf{Q}_Q(\mathbb{R}^n)$ as $n \geq 2$. Even in the case of $n = 1$, where a natural subtraction for any two members in $\mathbf{Q}_Q(\mathbb{R})$ does exist (see the Preliminaries), one still needs to be careful in perturbations of multiple-valued functions so that Definition 2 is fulfilled. A simple example is to naively define a family of multiple-valued functions from subtraction between a $\mathbf{Q}_2(\mathbb{R})$-valued function $f(x) := [\![x]\!] + [\![-x]\!]$, where $x \in (-1, 1)$, and the member $[\![-1]\!] + [\![1]\!]$ by $f^t(x) := [\![(1-t)x + t(1-x)]\!] + [\![-(1-t)x + t(1+x)]\!]$. As $t > 0$, it is easy to verify that there is no point with multiplicity 2 remained in the value of the continuous function f^t. According to Definition 2, one only allows (parametrized) diffeomorphisms (or bi-Lipschitz homeomorphisms) $\psi_x := \psi(x, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$, where $\psi \in C_c^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, in the range-variations. Hence, the point with multiplicity 2 of the continuous $\mathbf{Q}_2(\mathbb{R})$-valued function f should not disappear in the class of range-variations given in Definition 2. Therefore, the family of perturbed $\mathbf{Q}_2(\mathbb{R})$-valued functions f^t can't be generated from the definition of range-variations.

To overcome the difficulty in range-variations, we give a method to build up the so-called *nested admissible* closed balls (see Definition 6) for arbitrarily chosen member in $\mathbf{Q}_Q(\mathbb{R}^n)$. Namely, for any member $q \in \mathbf{Q}_Q(\mathbb{R}^n)$, there exists a sequence $q = q^{(0)}, q^{(1)}, ..., q^{(L)} = Q[\![a]\!]$ for some $L \in \mathbb{Z}_+$ and $a \in \mathbb{R}^n$ with strictly decreasing $\operatorname{card}\left(\operatorname{spt}\left(q^{(i)}\right)\right)$ as i increases such that

$$\cdots \subset\subset \mathbb{B}^{\mathbf{Q}}_{\rho_i}(q^{(i)}) \subset\subset \mathbb{B}^{\mathbf{Q}}_{\sigma_i}(q^{(i)}) \subset\subset \mathbb{B}^{\mathbf{Q}}_{\rho_{i+1}}(q^{(i+1)}) \subset\subset \cdots$$

for some sequence

$$0 = \rho_0 < \sigma_0 < \rho_1 < \sigma_1 < \cdots < \rho_L < \sigma_L := \infty.$$

One observes that there is a natural subtraction between this arbitrarily chosen member q and any member in its *admissible* closed balls (or neighborhoods). Then we show that the monotonicity formula can be established by the perturbations associated with this type of *admissible* balls on $\mathbb{B}^{\mathbf{Q}}_\rho(q^{(i)})$ (see Definition 5), where ρ is roughly between ρ_i and σ_i. The monotonicity formula would be used to bound $\sigma_i - \rho_i$ from above by Dirichlet integral of $(\boldsymbol{\xi}_0 \circ f, h)$ for each i. Note that $\rho_{i+1} - \sigma_i$ is bounded from above by a constant depending only on n and Q (see Proposition 3). This allows us to establish a “global” monotonicity formula in the proof of Lemma 3 in Step 3. Here, “global” means that the formula is not restricted to the radius

of admissible closed ball that one derives the formula but extensible to a bigger admissible closed ball (with changes of some constants depending only on n and Q).

Once we establish the global monotonicity formula, the proof of interior continuity follows Grüter's argument in [8], which is based on an argument by contradiction. Namely, suppose f is not continuous at $w \in \Omega$, we may assume that there exists a positive number $\tau_* > 0$ such that for any open ball $\mathbb{U}_R(w) \subset\subset \Omega$ there is always a point $w^* \in \mathbb{U}_r(w)$ fulfilling the inequality, $\mathrm{dist}_{\mathbf{Q}_Q(\mathbb{R}^n)} \left(f(w^*), f_{|\partial \mathbb{U}_r(w)} \right) > \tau_*$, for some $r \in (R/2, R)$. From Courant-Lebesgue Lemma (see Lemma 2) and the bi-Lipschitz embedding $\boldsymbol{\xi} : \mathbf{Q}_Q(\mathbb{R}^n) \to \mathbb{R}^N$, we may always choose a good slice to have $\operatorname*{osc}_{\partial \mathbb{U}_r(w)} \boldsymbol{\xi} \circ f \leq C_1(Q, N) \cdot Dir(f : \mathbb{U}_R(w))^{1/2}$ for some arbitrary small $r > 0$ (by choosing $R > 0$ sufficiently small). Then we apply the type of Grüter's monotonicity formula in [8] to show that $\tau_* \leq C_2(Q, N) \cdot Dir(G : \mathbb{U}_R(w))^{1/2}$. Finally, we show in the proof that both of these two Dirichlet integrals are bounded from above by $\alpha(R) > 0$, which tends to zero as $R \to 0$ and is independent of the choice of $\boldsymbol{\xi}_0$.

The rest of this article is arranged as the following. In Section 2, we collect some notations and results from [1], [6] and [8] to keep this article short and self-contained. Our main Theorem and its proof is contained in Section 3.

2. Preliminaries

For any single point $y \in \mathbb{R}^n$, let $[\![y]\!]$ denote the zero dimensional integral current, $[\![y]\!] : g \mapsto g(y)$, where $g : \mathbb{R}^n \to \mathbb{R}$ is any continuous test function with compact support. For a given positive integer Q, let $\mathbf{Q}_Q(\mathbb{R}^n) := \{ \sum_{i=1}^{Q} [\![p_i]\!] : p_i \in \mathbb{R}^n \}$, where p_i, p_j are not necessarily distinct as $i \neq j$. For our convenience, as we use the notation $p = \sum_{j=1}^{J} \ell_j [\![p_j]\!]$ for any member in $\mathbf{Q}_Q(\mathbb{R}^n)$, we mean $p_1, ..., p_J$ are distinct points in $\mathbb{R}^n$, ℓ_j is the multiplicity of p_j for each j and hence $\sum_{j=1}^{J} \ell_j = Q$. In [1], $\mathbf{Q}_Q(\mathbb{R}^n)$ is shown to be a metric space by associated with proper distance functions. One is the so-called flat metric, see [1, Chap.1]. The other one corresponding to the standard Euclidean distance function is given by

$$\mathcal{G}(p, q) := \inf \left\{ \left(\sum_{i=1}^{Q} \left| p_i - q_{\sigma(i)} \right|^2 \right)^{1/2} : \sigma \text{ is a permutation of } \{1, ..., Q\} \right\}$$

where $p = \sum_{i=1}^{Q} [\![p_i]\!]$ and $q = \sum_{j=1}^{Q} [\![q_j]\!]$. As $Q \geq 2$ and $\Omega \subset \mathbb{R}^m$ is an open set, $f : \Omega \subset \mathbb{R}^m \to \mathbf{Q}_Q(\mathbb{R}^n)$ is called a `multiple-valued function` or more precisely

$\mathbf{Q}_Q(\mathbb{R}^n)$-`valued function`, usually denoted as $f(x) = \sum_{i=1}^{Q} [\![f_i(x)]\!]$. Hence the support of $f(x)$, $\operatorname{spt}(f(x))$, is consisted of Q unordered points in $\mathbb{R}^n$ for all $x \in \Omega$. In fact, Almgren showed in [1, Section 1.1] and [1, Section 1.2] that the metric space $\mathbf{Q}_Q(\mathbb{R}^n)$ is bi-Lipschitz correspondence with a nQ-dimensional polyhedral cone $\mathbf{Q}^*$ in a Euclidean space $\mathbb{R}^{P(n,Q)\cdot Q}$. More precisely, there exists a positive integer $P(n,Q) > n$ such that for each fixed $\alpha = 1, ..., P(n,Q)$, there corresponds a straight line L_α in $\mathbb{R}^n$ and an orthogonal projection, denoted as $\Pi_\alpha \in O^*(n,1)$, into the α-th straight line in $\mathbb{R}^n$. For each $\alpha \in \{1, 2, ..., n\}$, L_α is chosen to be the coordinate axis of $\mathbb{R}^n$. Hence, for any $y = (y^1, ..., y^n) \in \mathbb{R}^n$ and $\alpha \in \{1, ..., n\}$, we have $\Pi_\alpha(y) = y^\alpha \in \mathbb{R}$. Thus, for each $\alpha \in \{1, ..., n\}$, Π_α induces the map $(\Pi_\alpha)_\#(\cdot) : \mathbf{Q}_Q(\mathbb{R}^n) \to \mathbf{Q}_Q(\mathbb{R})$ defined by

$$(\Pi_\alpha)_\# \left(\sum_{j=1}^{Q} [\![q_j]\!] \right) = \sum_{j=1}^{Q} [\![q_j^\alpha]\!] \tag{2.1}$$

where q_j^α is the α-th component of $q_j \in \mathbb{R}^n$, and

$$\xi_\alpha(\cdot) := \xi(\Pi_\alpha, \cdot) : \mathbf{Q}_Q(\mathbb{R}^n) \to \{(s_1, s_2, \cdots, s_Q) : s_1 \le s_2 \le \cdots \le s_Q\} \subset \mathbb{R}^Q. \tag{2.2}$$

Hence,

$$\xi_\alpha(q) = (q^\alpha_{\sigma(1)}, \cdots, q^\alpha_{\sigma(Q)}),$$

for some $\sigma \in \mathcal{P}_Q$: the permutation group of $\{1, ..., Q\}$. Note that, from (2.2), it is easy to verify

$$|\xi_\alpha(p) - \xi_\alpha(q)| = |\xi(\Pi_\alpha, p) - \xi(\Pi_\alpha, q)| = \mathcal{G}((\Pi_\alpha)_\#(p), (\Pi_\alpha)_\#(q)). \tag{2.3}$$

For each fixed coordinates of $\mathbb{R}^n$, Almgren introduce the Lipschitz correspondence

$$\boldsymbol{\xi}_0(\cdot) := (\xi(\Pi_1, \cdot) \cdots \xi(\Pi_n, \cdot)) : \mathbf{Q}_Q(\mathbb{R}^n) \to \mathbb{R}^{nQ} \tag{2.4}$$

with $Lip(\boldsymbol{\xi}_0) = 1$. The reader may verify from (2.3) that

$$|\boldsymbol{\xi}_0(p) - \boldsymbol{\xi}_0(q)|^2 = \sum_{\alpha=1}^{n} |\xi_\alpha(p) - \xi_\alpha(q)|^2 \le \mathcal{G}(p, q)^2 .$$

Moreover, $\boldsymbol{\xi}_0$ is not injective unless $n = 1$ (or see [1, Theorem 1.2]). However, by introducing more distinct orthogonal projections into the straight lines L_α, $\alpha \in \{n+1, ..., P(n,Q)\}$ (as defined in (2.2)), Almgren showed that

$$\boldsymbol{\xi}(\cdot) := \xi(\Pi_1, \cdot) \bowtie \cdots \bowtie \xi(\Pi_P, \cdot) : \mathbf{Q}_Q(\mathbb{R}^n) \to \mathbf{Q}^* \subset \mathbb{R}^N$$

is then a bi-Lipschitz correspondence, where $\mathbf{Q}^* := \boldsymbol{\xi}\left(\mathbf{Q}_Q(\mathbb{R}^n)\right)$ and $N = P(n,Q)Q$. Furthermore, both $Lip(\boldsymbol{\xi})$ and $Lip(\boldsymbol{\xi}^{-1})$ are positive numbers depending only on n and Q.

Besides the bi-Lipschitz correspondence $\boldsymbol{\xi}$ introduced in [1], there is a modified bi-Lipschitz and locally (or infinitesimally) equidistant correspondence introduced by Brian White. The reader could find it in literature from the article of Sheldon

Chang (see [2, p. 706]) or from [4] for more details. The modified correspondence is constructed by choosing $P = P(n,Q)$ distinct orthonormal coordinate basis (by rotating the orthonormal coordinates of $\mathbb{R}^n$), by taking the orthogonal projections $\Pi_1, ..., \Pi_{P\cdot n}$ (as done in [1, Chapter 1.2]) as a complete set of coordinate projections, and by rescaling the resulting $\boldsymbol{\xi}$ under a proper scaling factor depending on $P(n,Q)$.

Denote an affine map $A : \mathbb{R}^m \to \mathbb{R}^n$ by $A(x) = A(x_0) + L(x - x_0)$, where $L \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ is the linear part. Let $\mathrm{A}(m,n)$ denote the set of affine maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. As $A \in \mathrm{A}(m,n)$, we let

$$|A| := \left(\sum_{i=1}^{m} \ |D_i L|^2 \right)^{1/2} \in \mathbb{R}$$

where $D_i L = \frac{\partial L}{\partial x_i}$. A multiple-valued function $\mathcal{A} : \mathbb{R}^m \to \mathbf{Q}_Q(\mathbb{R}^n)$ is said to be `affine` if there are affine maps $A_1, ..., A_Q \in \mathrm{A}(m,n)$ such that $\mathcal{A} := \sum_{i=1}^{Q} [\![A_i]\!]$. Let

$$|\mathcal{A}| := \left(\sum_{i=1}^{Q} \ |A_i|^2 \right)^{1/2} . \tag{2.5}$$

Assume $\Omega \subset \mathbb{R}^m$ is an open set and $x_0 \in \Omega$, then $f : \Omega \to \mathbf{Q}_Q(\mathbb{R}^n)$ is called `approximately affinely approximatable` at $x_0 \in \Omega$ if there exists an affine function $\mathcal{A} : \mathbb{R}^m \to \mathbf{Q}_Q(\mathbb{R}^n)$ such that

$$\mathrm{ap} \lim_{x \to x_0} \frac{\mathcal{G}(f(x), \mathcal{A}(x))}{|\ x - x_0 \ |} = 0.$$

Such multiple-valued function $\mathcal{A}$ is uniquely determined and denoted by ap $Af(x_0)$. Hence as $f : \Omega \to \mathbf{Q}_Q(\mathbb{R}^n)$ is approximately affinely approximatable at $x_0 \in \Omega$, we write

$$\mathrm{ap}\ Af(x_0) = \sum_{i=1}^{Q} [\![A_i(x_0)]\!].$$

In [1, Theorem 1.4 (3)], Almgren proved that if $\boldsymbol{\xi} \circ f : \Omega \to \mathbb{R}^N$ is approximately differentiable at $x_0 \in \Omega$ for a multiple-valued function $f(x) = \sum_{i=1}^{Q} [\![f_i(x)]\!] \in \mathbf{Q}_Q(\mathbb{R}^n)$, then f is `strongly approximately affinely approximatable` at x_0. In other words, if $\boldsymbol{\xi} \circ f$ is approximate differentiable at x_0, then $f_i(x_0) = f_j(x_0)$ implies that

$$\mathrm{ap}\ D_x f_i(x_0) = \mathrm{ap}\ D_x f_j(x_0).$$

Furthermore, ap $Af(x_0)$ is uniquely determined by $f(x_0)$ and ap $D(\boldsymbol{\xi} \circ f)(x_0)$ and

$$|\mathrm{ap}\ Af(x_0)| = |\mathrm{ap}\ D(\boldsymbol{\xi}_0 \circ f)(x_0)|.$$

In [1], Almgren also introduced the space, $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ (still called Sobolev spaces for our convenience), for the class of multiple-valued functions $f : \Omega \subset \mathbb{R}^m \to \mathbf{Q}_Q(\mathbb{R}^n)$. Recall that one usually uses $W^{1,2}(\Omega, \mathbb{R}^N)$ denote the Sobolev space of $\mathbb{R}^N$-valued functions defined on Ω with their first order distributional partial derivatives

being $\mathcal{L}^m$ square summable over Ω. A function $f \in W^{1,2}(\Omega, \mathbb{R}^N)$ is said to be *strictly defined* if $f(x) = y$ as $x \in \Omega$, $y \in \mathbb{R}^N$, and

$$\lim_{r \to 0} r^{-m} \int_{\mathbb{U}_r^m(x)} |f(z) - y| \; d\mathcal{L}^m z = 0.$$

In fact, any $f \in W^{1,2}(\Omega, \mathbb{R}^N)$ agrees with a strictly defined $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ $\mathcal{L}^m$ almost everywhere on Ω (see [1, Appendix 1.2] or [11, p. 592]). In [1], the Sobolev space for multiple-valued functions is defined by

$$\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n)) := \{ f : \Omega \subset \mathbb{R}^m \to \mathbf{Q}_Q(\mathbb{R}^n) : \boldsymbol{\xi} \circ f \in W^{1,2}(\Omega, \mathbb{R}^N) \}. \tag{2.6}$$

We say that f is *strictly defined* if $\boldsymbol{\xi} \circ f$ is strictly defined. Suppose $f \in \mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$, then by [1, Theorem 2.2],

$$|\text{ap } D\,(\boldsymbol{\xi}_0 \circ f(x))| = |\text{ap } Af(x)| \tag{2.7}$$

a.e. $x \in \Omega$. From [1, Definition 2.1] and [1, Theorem 2.2], the Dirichlet integral of a multiple-valued function $f \in \mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ over an open set Ω is given by

$$Dir(f; \Omega) = \int_\Omega |\text{ap } Af(x)|^2 \; d\mathcal{L}^m x = \int_\Omega |\text{ap } D\,(\boldsymbol{\xi}_0 \circ f(x))|^2 \; d\mathcal{L}^m x. \tag{2.8}$$

Note that, since the norm of any affine map defined in (2.5) is independent of the choice of coordinates of $\mathbb{R}^n$, we have

$$|\text{ap } D(\boldsymbol{\xi}_0 \circ f)| = |\text{ap } D(\widetilde{\boldsymbol{\xi}_0} \circ f)| \tag{2.9}$$

for any two distinct Lipschitz correspondences $\boldsymbol{\xi}_0$ and $\widetilde{\boldsymbol{\xi}_0}$. Hence, the Dirichlet integral in (2.8) is independent of the choice of orthonormal coordinates in $\mathbb{R}^n$.

Below, we recall from Federer [6] and Grüter [8] some properties of functions in Sobolev spaces. A map $X : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^N$ is called approximately differentiable at $w_0 \in \Omega$ with the approximate differential $\nabla X(w_0)$, if there exists $X_0 \in \mathbb{R}^N$ such that for every $\epsilon > 0$

$$\Theta^2 \left(\mathcal{L}^2 \lfloor \Omega \setminus \{ w : |X(w) - X(w_0) - \nabla X(w_0)(w - w_0)| \le \epsilon \cdot |w - w_0| \}, w_0 \right) = 0$$

where Θ^2 denotes for the two-dimensional density and $\mathcal{L}^2 \lfloor D$ indicates the Lebesgue measure restricted to a set D (see Federer [6, 2.10.19] or Grüter [8, Definition 2.2]). Note, here is another characterization of approximate differentiability: $X \in \Omega \subset \mathbb{R}^2 \to \mathbb{R}^N$ is approximately differentiable at $w_0 \in \Omega$ with the approximate differential $\nabla X(w_0)$, if and only if there exists a neighborhood U of w_0 and a map $Y : U \to \mathbb{R}^N$ such that Y is differentiable at w_0 and

$$\Theta^2 \left(\mathcal{L}^2 \lfloor \Omega \setminus \{ w : X(w) \ne Y(w) \}, w_0 \right) = 0.$$

The approximate differential is $\nabla Y(w_0)$. If $X \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^N)$, then X is approximately differentiable almost everywhere and the weak derivative coincides with the approximate differential almost everywhere (see Federer [6, Theorem 4.5.9 (26), (30) (VI))].

Definition 1 (the set of "good" points of a function in the Sobolev space $W^{1,2}(\Omega)$, cf. Grüter [8])**.** *Suppose Ω is a domain in $\mathbb{R}^2$ and $\mathcal{M}$ is a complete Riemannian submanifolds in $\mathbb{R}^N$ for some positive integer N. Let $e(X)(w) := |\nabla X|^2(w)$ be the energy density of X. Define the set of "good" points of $X \in W^{1,2}(\Omega, \mathbb{R}^N)$ by*

$$
\begin{aligned}
A := \quad & \{w \in \Omega : X \text{ is approximately differentiable at } w, \\
& w \text{ is a Lebesgue point of } e(X),\ e(X)(w) \neq 0\ \}.
\end{aligned}
$$

Below, we collect some lemmas from [8]. We denote by $\mathbb{U}_r(w_0) \subset \mathbb{R}^2$ the open ball $\{x \in \mathbb{R}^2 : |x - w_0| < r\}$ and $\mathbb{B}_r(w_0) \subset \mathbb{R}^2$ the closed ball $\{x \in \mathbb{R}^2 : |x - w_0| \leq r\}$.

Lemma 1 (Grüter [8, 2.5])**.** *Let $X \in W^{1,2}(\Omega, \mathbb{R}^N)$ satisfy the conformal conditions,*

$$|X_u|^2 = |X_v|^2,\ X_u \cdot X_v = 0,\ a.e.\ in\ \Omega.$$

Suppose Ω is open and $w^* \in A \cap \Omega$. Then,

$$\limsup_{\sigma \to 0} \sigma^{-2} \int_{\Omega \cap \{w : |X(w) - X(w^*)| < \sigma\}} |\nabla X|^2 \geq 2\pi.$$

Lemma 2 (Courant-Lebesgue Lemma, Grüter [8, 2.6])**.** *There is a constant $C = C(N) > 0$ with the following property. For any open set $\Omega \subset \mathbb{R}^2$, any $X \in W^{1,2}(\Omega, \mathbb{R}^N)$, any $w_0 \in \Omega$, and any $0 < R < dist(w_0, \partial\Omega)$, there exists $r \in [\frac{1}{2}R, R]$ such that*

$$\underset{\partial \mathbb{U}_r(w_0)}{osc} X \leq C(N) \cdot \left(\int_{\mathbb{U}_R(w_0)} |\nabla X|^2 \right)^{1/2}.$$

3. The interior continuity

In Definition 2, we define the class of weakly harmonic multiple-valued functions in the Sobolev space $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$. The perturbations in Definition 2 are induced from the `range-variations`, which are also called outer variations in [4].

Definition 2 (weakly harmonic multiple-valued functions)**.** *Let $\Omega \subset \mathbb{R}^m$ be an open set and $\epsilon > 0$ be sufficiently small. For any given $\psi \in C_c^\infty(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ such that the support $spt(\psi) \subset \Omega' \times \mathbb{R}^n$ for some $\Omega' \subset\subset \Omega$, define the induced perturbation of f by*

$$f^t(x) := \sum_{i=1}^{Q} [\![f_i(x) + t \cdot \psi(x, f_i(x))]\!] \tag{3.1}$$

where $t \in (-\epsilon, \epsilon)$. Then we say that $f \in \mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ is `weakly harmonic` *if and only if*

$$\lim_{t \to 0} \frac{Dir(f^t; \Omega) - Dir(f; \Omega)}{t} = 0. \tag{3.2}$$

In Definition 3, we define the class of weakly Noether harmonic multiple-valued functions in the Sobolev space $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ of multiple-valued functions. The perturbations in Definition 3 are induced from the `domain-variations`, which are also called inner variations in [4].

Definition 3 (weakly Noether harmonic multiple-valued functions). *Let $\Omega \subset \mathbb{R}^m$ be an open set and $\epsilon > 0$ be sufficiently small. Assume that, for any given $\phi \in C_c^\infty(\Omega, \mathbb{R}^m)$ and any fixed $t \in (-\epsilon, \epsilon)$, $X^t : \Omega \to \Omega$ defined by $X^t(x) := x + t \cdot \phi(x)$ is a diffeomorphism of Ω, leaving the boundary $\partial\Omega$ fixed. We say that $f \in \mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ is* ***weakly Noether harmonic*** *if and only if*

$$\lim_{t \to 0} \frac{Dir(f \circ X^t; \Omega) - Dir(f; \Omega)}{t} = 0.$$

Definition 4 (weakly stationary-harmonic multiple-valued functions). *We say that $f \in \mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ is* ***weakly stationary-harmonic*** *if and only if f is* ***weakly harmonic*** *and* ***weakly Noether harmonic****.*

The main goal of this article is to prove the interior continuity of any two-dimensional weakly stationary-harmonic multiple-valued function in the Sobolev space $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ of multiple-valued functions.

Theorem 1. *Suppose $f \in \mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ is a strictly defined multiple-valued function, where Ω is a simply connected open subset in $\mathbb{R}^2$. Then f is continuous in the interior of Ω as f is weakly stationary-harmonic with finite Dirichlet integral over Ω.*

3.1. **The domain-variations:** For convenience, we identify $\mathbb{C}$ with $\mathbb{R}^2$ below. Note that Proposition 1 is a well-known result in the theory of 2-dimensional harmonic mappings into Riemannian manifolds.

Proposition 1. *Denote by $\mathbb{U}_{R_0}(0) \subset \mathbb{C}$ the open ball of radius $R_0 > 0$ with center at the origin of complex plane $\mathbb{C}$. Suppose $f \in \mathcal{Y}_2(\mathbb{U}_{R_0}(0), \mathbf{Q}_Q(\mathbb{R}^n))$ is weakly stationary-harmonic. Then,*

(1) *The* ***Hopf differential*** *of $\boldsymbol{\xi}_0 \circ f \in W^{1,2}(\mathbb{U}_{R_0}(0), \mathbb{R}^{nQ})$,*

$$\Phi(z) := \left[\left(\left| \tfrac{\partial(\boldsymbol{\xi}_0 \circ f)}{\partial u} \right|^2 - \left| \tfrac{\partial(\boldsymbol{\xi}_0 \circ f)}{\partial v} \right|^2 \right) - 2i \left\langle \tfrac{\partial(\boldsymbol{\xi}_0 \circ f)}{\partial u}, \tfrac{\partial(\boldsymbol{\xi}_0 \circ f)}{\partial v} \right\rangle \right] dz^2 =: \varphi(z)\ dz^2$$

is holomorphic in the interior of $\mathbb{U}_{R_0}(0)$. Here, $z = u + iv$ is a complex variable and $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors in Euclidean spaces.

(2) *There exists a harmonic function $h : \mathbb{U}_{R_0}(0) \to \mathbb{R}^2 \cong \mathbb{C}$ fulfilling*

$$\begin{cases} \frac{\partial}{\partial z} h = \frac{-1}{4}\varphi, \\ \frac{\partial}{\partial \bar{z}} h = 1, \end{cases} \tag{3.3}$$

such that

$$(\boldsymbol{\xi}_0 \circ f, h) \in W^{1,2}(\mathbb{U}_{R_0}(0), \mathbb{R}^{nQ} \times \mathbb{R}^2)$$

is weakly conformal in $\mathbb{U}_{R_0}(0)$*, i.e.,*

$$\left|\frac{\partial(\boldsymbol{\xi}_0\circ f,h)}{\partial u}\right| = \left|\frac{\partial(\boldsymbol{\xi}_0\circ f,h)}{\partial v}\right| \text{ and } \left\langle \frac{\partial(\boldsymbol{\xi}_0\circ f,h)}{\partial u}, \frac{\partial(\boldsymbol{\xi}_0\circ f,h)}{\partial v}\right\rangle = 0, \text{ a.e. in } \mathbb{U}_{R_0}(0).$$

(3) *Suppose* $\boldsymbol{\xi}_0$, $\widetilde{\boldsymbol{\xi}}_0$ *are two distinct Lipschitz correspondences and* φ, $\widetilde{\varphi}$ *are the induced holomorphic functions defined in (1) respectively. Then,* $|\varphi - \widetilde{\varphi}| = |\varphi_2 - \widetilde{\varphi}_2| = C(\boldsymbol{\xi}_0, \widetilde{\boldsymbol{\xi}}_0)$ *is a constant depending only on the choice of* $\boldsymbol{\xi}_0$ *and* $\widetilde{\boldsymbol{\xi}}_0$*. Moreover,*

$$C(\boldsymbol{\xi}_0, \widetilde{\boldsymbol{\xi}}_0) \leq \frac{4}{\pi R_0^2} \cdot Dir(f; \mathbb{U}_{R_0}(0)). \tag{3.4}$$

Proof. (1) We sketch the proof from [14]. For any smooth function $\eta : \mathbb{U}_{R_0}(0) \to \mathbb{R}$ with compact support, let $X^t(u,v) = (u + t\cdot\eta(u,v), v)$. By the chain rule for weak derivatives, we have

$$\begin{cases} \frac{\partial\left(\boldsymbol{\xi}_0\circ f\circ X^t(u,v)\right)}{\partial u} = \frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial u}\left(X^t(u,v)\right)\cdot\left(1 + t\cdot\frac{\partial\eta}{\partial u}\right), \\ \frac{\partial\left(\boldsymbol{\xi}_0\circ f\circ X^t(u,v)\right)}{\partial v} = \frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial u}\left(X^t(u,v)\right)\cdot\left(t\cdot\frac{\partial\eta}{\partial v}\right) + \frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial v}\left(X^t(u,v)\right). \end{cases}$$

From the change of variables, $(\sigma,\tau) = X^t(u,v)$, and the definition of Dirichlet integral of multiple-valued functions in (2.8), we have

$$Dir(f^t;\Omega) = \int_\Omega \left[\left|\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial\sigma}\right|^2\cdot\left(1 + t\frac{\partial\eta}{\partial u}\right)^2 + \left|\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial\sigma}\cdot\left(t\frac{\partial\eta}{\partial v}\right) + \frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial\tau}\right|^2\right] \frac{dudv}{1+t\cdot\eta_u}.$$

Then, the vanishing of the first variations of the Dirichlet integral of f with respect to the domain-variations X^t gives

$$\int_\Omega \left[\left(\left|\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial u}\right|^2 - \left|\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial v}\right|^2\right)\frac{\partial\eta}{\partial u} + 2\left\langle\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial u}, \frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial v}\right\rangle\frac{\partial\eta}{\partial v}\right] dudv = 0.$$

A similar argument, using the diffeomorphism $X^t(u,v) = (u, v + t\cdot\zeta(u,v))$, gives

$$\int_\Omega \left[\left(\left|\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial u}\right|^2 - \left|\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial v}\right|^2\right)\frac{\partial\zeta}{\partial v} - 2\left\langle\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial u}, \frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial v}\right\rangle\frac{\partial\zeta}{\partial u}\right] dudv = 0.$$

These two equations provide the weak form of the Cauchy-Riemann equations for the L^1-function

$$\varphi(z) = \left(\left|\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial u}\right|^2 - \left|\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial v}\right|^2\right) - 2i\left\langle\frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial u}, \frac{\partial(\boldsymbol{\xi}_0\circ f)}{\partial v}\right\rangle. \tag{3.5}$$

By Weyl's lemma, φ is a holomorphic function of z.

(2) If Φ is identically zero, then $\boldsymbol{\xi}_0 \circ f$ is weakly conformal. From (3.3), the assertion is then proved by choosing $h = \overline{z}$ (i.e., $h = u - iv$). Hence, we assume below that Φ is not identically zero. Below we would like to follow the trick in [9] to construct a $\mathbb{R}^2$-valued harmonic function h showing that the Hopf differential of

$$(\boldsymbol{\xi}_0 \circ f, h) : \mathbb{U}_{R_0}(0) \to \mathbb{R}^{nQ} \times \mathbb{R}^2$$

is identically zero.

For convenience, we abuse the notation by writing $h = (h_1, h_2) \in \mathbb{R}^2$ as $h = h_1 + i \cdot h_2 \in \mathbb{C}$ below. Let $\Phi_h(z)$ denote the Hopf differential of $(\boldsymbol{\xi}_0 \circ f, h)$. By a simple computation, one can verify

$$\Phi_h(z) = \Phi(z) + \left[\left(\left|\tfrac{\partial h}{\partial u}\right|^2 - \left|\tfrac{\partial h}{\partial v}\right|^2\right) - 2i\left\langle \tfrac{\partial h}{\partial u}, \tfrac{\partial h}{\partial v}\right\rangle\right]\, dz^2 \tag{3.6}$$

$$= \left[\varphi(z) + 4\, \tfrac{\partial h}{\partial z}\, \tfrac{\partial \overline{h}}{\partial z}(z)\right]\, dz^2.$$

Since we know that φ is holomorphic, there exists a holomorphic function ψ satisfying $\psi' = \frac{-1}{4}\varphi$. Let

$$h(z) = \psi(z) + \bar{z}. \tag{3.7}$$

Then, it is easy to verify that both h_1 and h_2 are real-valued harmonic functions. Moreover, a simple calculation shows that h also fulfills

$$\tfrac{\partial h}{\partial z}\, \tfrac{\partial \overline{h}}{\partial z} = -\tfrac{1}{4}\varphi. \tag{3.8}$$

From (3.6) and (3.8), $\Phi_h(z) \equiv 0$. Hence, one concludes that $(\boldsymbol{\xi}_0 \circ f, h) : \mathbb{U}_{R_0}(0) \to \mathbb{R}^{nQ} \times \mathbb{R}^2$ is weakly conformal.

(3) Let $\varphi = \varphi_1 + i \cdot \varphi_2$ and $\widetilde{\varphi} = \widetilde{\varphi}_1 + i \cdot \widetilde{\varphi}_2$. Notice that, from (2.9), we have

$$\begin{cases} |\partial_u(\boldsymbol{\xi}_0 \circ f)| = |\partial_u(\widetilde{\boldsymbol{\xi}}_0 \circ f)|, \\ |\partial_v(\boldsymbol{\xi}_0 \circ f)| = |\partial_v(\widetilde{\boldsymbol{\xi}}_0 \circ f)|. \end{cases}$$

Hence, from (3.5), $\widetilde{\varphi}_1 = \varphi_1$. Since both φ and $\widetilde{\varphi}$ are holomorphic, they satisfy the Cauchy-Riemann equations. Thus,

$$\begin{cases} \partial_u(\widetilde{\varphi}_2 - \varphi_2) = -\partial_v(\widetilde{\varphi}_1 - \varphi_1) = 0, \\ \partial_v(\widetilde{\varphi}_2 - \varphi_2) = \partial_u(\widetilde{\varphi}_1 - \varphi_1) = 0. \end{cases}$$

Therefore, one concludes that $|\widetilde{\varphi} - \varphi| = |\widetilde{\varphi}_2 - \varphi_2| = C(\boldsymbol{\xi}_0, \widetilde{\boldsymbol{\xi}}_0)$ is a constant depending on the choice of $\boldsymbol{\xi}_0$ and $\widetilde{\boldsymbol{\xi}}_0$ (or the choice of coordinates of $\mathbb{R}^n$). Note that, from (3.5), we have

$$|\varphi| \le 2 \cdot |\nabla(\boldsymbol{\xi}_0 \circ f)|^2;\ |\widetilde{\varphi}| \le 2 \cdot |\nabla(\widetilde{\boldsymbol{\xi}}_0 \circ f)|^2. \tag{3.9}$$

Hence,

$$C(\boldsymbol{\xi}_0, \widetilde{\boldsymbol{\xi}}_0) = |\widetilde{\varphi} - \varphi| \le |\widetilde{\varphi}| + |\varphi| \le 2 \cdot |\nabla(\boldsymbol{\xi}_0 \circ f)|^2 + 2 \cdot |\nabla(\widetilde{\boldsymbol{\xi}}_0 \circ f)|^2. \tag{3.10}$$

From integration over the set $\mathbb{U}_{R_0}(0)$, (3.10) gives

$$C(\boldsymbol{\xi}_0, \widetilde{\boldsymbol{\xi}}_0) \le \frac{4}{\pi R_0^2} \cdot Dir(f; \mathbb{U}_{R_0}(0)).$$

□

3.2. **The range-variations:** We would like to follow the proof of Theorem 3.10 in Grüter's paper [8] to derive the key estimates. In order to apply Grüter's argument, we build up the so-called *nested admissible* closed balls of a member $q \in \mathbf{Q}_Q(\mathbb{R}^n)$ (see Definition 6) for constructing the admissible range-variations. For this purpose, we need Proposition 2 and Proposition 3.

Proposition 2. *Suppose $n \geq 2$, $Q \geq 2$. Then, there exists a a positive number*

$$0 < \theta_0 = \theta_0(n, Q) \leq \frac{\pi}{4}$$

such that for any $q = \sum_{j=1}^{Q} [\![q_j]\!] \in \mathbf{Q}_Q(\mathbb{R}^n)$ one could find a coordinate basis of $\mathbb{R}^n$, denoted by $\{e_1, ..., e_n\}$, fulfilling the inequality

$$\left| \left\langle e_\alpha, \frac{q_i - q_j}{|q_i - q_j|} \right\rangle \right| \geq \sin\theta_0 \tag{3.11}$$

for any $\alpha \in \{1, ..., n\}$ and any $q_i \neq q_j$.

Proof. For any non-zero vector $q_i - q_j$, let

$$v_{i,j} := \frac{q_i - q_j}{|q_i - q_j|} = \sum_{\alpha=1}^{n} c_{i,j}^{\alpha} \cdot e_\alpha$$

where $c_{i,j}^{\alpha} = \sin\measuredangle(v_{i,j}, \pi_\alpha)$ and $\measuredangle(v_{i,j}, \pi_\alpha)$ is the positive angle between $v_{i,j}$ and its orthogonal projection into the hyperplane π_α. Note here, we let $\measuredangle(v_{i,j}, \pi_\alpha) \in [0, \frac{\pi}{2}]$ and $\measuredangle(v_{i,j}, \pi_\alpha) = \frac{\pi}{2}$ when the projection of $v_{i,j}$ is null. The proof is equivalent to showing that there exist a positive number $\theta_0 = \theta_0(n, Q) > 0$ and a coordinate basis of $\mathbb{R}^n$ such that the geodesic distance on $\mathbb{S}^{n-1}(1)$ between any member in $\bigcup_{i,j\in\{1,...,Q\}} \left\{ \frac{q_i - q_j}{|q_i - q_j|} : q_i \neq q_j \right\}$ and any member in $\bigcup_{\alpha=1}^{n} \pi_\alpha \cap \mathbb{S}^{n-1}(1)$ is no less than $\theta_0(n, Q)$. In other words, we may choose a new coordinate basis of $\mathbb{R}^n$ such that $\measuredangle(v_{i,j}, \pi_\alpha) \geq \theta_0(n, Q) > 0$ for all non-zero $v_{i,j}$ and $\alpha \in \{1, ..., n\}$, where π_α is defined with respective to the new coordinate basis of $\mathbb{R}^n$. Notice that the geodesic distance on the unit sphere between v_k and $\pi_\alpha \cap \mathbb{S}^{n-1}(1)$ is nothing else but the positive angle between $v_k \in \mathbb{R}^n$ and $u_\alpha \in \pi_\alpha \cap \mathbb{S}^{n-1}(1)$.

We may denote by

$$\{v_1, ..., v_\ell\} = \bigcup_{i,j\in\{1,...,Q\}} \left\{ \frac{q_i - q_j}{|q_i - q_j|} : q_i \neq q_j \right\}$$

for some $\ell \leq Q(Q-1)$, where $v_i \neq v_j$ for any distinct $i, j \in \{1, ..., \ell\}$. Let

$$\delta_\ell := \sin^{-1}\left(\frac{1}{\sqrt{n}}\right) \cdot \left(\frac{1}{2}\right)^{(n-1)(\ell-1)}. \tag{3.12}$$

Namely, we let $\delta_{\ell+1} = \delta_\ell \cdot \left(\frac{1}{2}\right)^{n-1}$ and $\delta_1 = \sin^{-1}\left(\frac{1}{\sqrt{n}}\right)$. The proof is an induction argument.

For any given one point $v_1 \in \mathbb{S}^{n-1}(1)$, we choose a new coordinate basis of $\mathbb{R}^n$ by rotation of the original one so that $v_1 = \sum_{\alpha=1}^{n} \frac{1}{\sqrt{n}} \cdot e^1_\alpha$, where $\{e^1_1, ..., e^1_n\}$ is the new coordinate basis. In other words, $v_1 = \left(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}\right)$ under the new coordinates, and $\measuredangle(v_1, \pi_\alpha) = \sin^{-1}\left(\frac{1}{\sqrt{n}}\right)$ for all $\alpha \in \{1, ..., n\}$. Hence the proof is done in this case.

Assume that, for any $\ell - 1$ distinct points $\{v_1, ..., v_{\ell-1}\} \subset \mathbb{S}^{n-1}(1)$ and $\ell \geq 2$, there exists a coordinate basis $\left\{e^{\ell-1}_1, ..., e^{\ell-1}_n\right\}$ such that

$$\measuredangle\left(v_b, \pi^{\ell-1}_\alpha\right) \geq \delta_{\ell-1}, \ \forall\ b \in \{1, ..., \ell-1\} \text{ and } \forall\ \alpha \in \{1, ..., n\} \tag{3.13}$$

where $\pi^{\ell-1}_\alpha$ is the hyperplane with normal vector $\pm e^{\ell-1}_\alpha$ under the coordinate basis $\left\{e^{\ell-1}_1, ..., e^{\ell-1}_n\right\}$. Then, for any extra given point $v_\ell \in \mathbb{S}^{n-1}(1)$, we would show below that there exists a new coordinate basis $\left\{e^\ell_1, ..., e^\ell_n\right\}$, derived from $\left\{e^{\ell-1}_1, ..., e^{\ell-1}_n\right\}$ by rotation, such that

$$\measuredangle\left(v_b, \pi^\ell_\alpha\right) \geq \delta_\ell, \ \forall\ b \in \{1, ..., \ell\} \text{ and } \forall\ \alpha \in \{1, ..., n\}.$$

We may assume that

$$\measuredangle\left(v_\ell, \pi^{\ell-1}_\alpha\right) < \delta_\ell \tag{3.14}$$

for at least one of $\alpha \in \{1, ..., n\}$. Otherwise, we can just choose the new coordinate basis by letting $e^\ell_i := e^{\ell-1}_i$ for all $i \in \{1, ..., n\}$. On the other hand, (3.14) can't hold for all hyperplane $\pi^{\ell-1}_\alpha$, $\alpha = 1, ..., n$. Otherwise, because $\delta_\ell \lneqq \sin^{-1}(\frac{1}{\sqrt{n}})$ for all $\ell \geq 2$, we conclude $|v_\ell| \gneqq 1$, contradicting $v_\ell \in \mathbb{S}^{n-1}(1)$. Hence, we may assume that

$$\measuredangle\left(v_\ell, \pi^{\ell-1}_{\alpha_1}\right) \leq \cdots \leq \measuredangle\left(v_\ell, \pi^{\ell-1}_{\alpha_k}\right) < \delta_{\ell-1} \leq \measuredangle\left(v_\ell, \pi^{\ell-1}_{\alpha_{k+1}}\right) \leq \cdots \leq \measuredangle\left(v_\ell, \pi^{\ell-1}_{\alpha_n}\right) \tag{3.15}$$

for some $k \in \{1, ..., n-1\}$. Note that rotating the coordinate basis of $\mathbb{R}^n$ so that the fixed point $v_\ell = \sum_{\alpha=1}^{n} c^\alpha_{\ell-1} \cdot e^{\ell-1}_\alpha$ can be re-written as $v_\ell = \sum_{\alpha=1}^{n} c^\alpha_\ell \cdot e^\ell_\alpha$ under the new coordinate basis $\{e^\ell_1, ..., e^\ell_n\}$ can be viewed as moving a point on $\mathbb{S}^{n-1}(1)$ from $\sum_{\alpha=1}^{n} c^\alpha_{\ell-1} \cdot e^{\ell-1}_\alpha$ to $\sum_{\alpha=1}^{n} c^\alpha_{\ell-1} \cdot e^{\ell-1}_\alpha$ along the shortest geodesic path under the same coordinate basis.

Below, we proceed the proof by viewing the rotation of coordinate basis as moving points on the unit sphere along geodesic.

We first move v_ℓ on the unit sphere along the shortest geodesic connecting v_ℓ and $\pm e^{\ell-1}_{\alpha_1}$, where $\pm e^{\ell-1}_{\alpha_1}$ is either $e^{\ell-1}_{\alpha_1}$ or $-e^{\ell-1}_{\alpha_1}$ depending on which one is shorter, until $\measuredangle\left(v_\ell, \pi^{\ell-1}_{\alpha_1}\right) = \frac{1}{2}\delta_{\ell-1}$. Denote by $v^-_\ell(1)$ the position of v_ℓ before this movement and $v^+_\ell(1)$ the new position of v_ℓ after this movement. Similarly, for each $i \in \{2, ..., n-1\}$, denote by $v^-_\ell(i)$ and $v^+_\ell(i)$ the position before and after the movement respectively. Note, $v^+_\ell(i) = v^-_\ell(i+1)$ for each i. For each $i \in \{2, ..., n-1\}$, we

move $v_\ell^-(i)$ with respective to $\pi_{\alpha_i}^{\ell-1}$ under this procedure so that $\measuredangle\left(v_\ell^+(i),\pi_{\alpha_i}^{\ell-1}\right)=\left(\frac{1}{2}\right)^i\delta_{\ell-1}$. Whenever $\measuredangle\left(v_\ell^-(i),\pi_{\alpha_i}^{\ell-1}\right)\geq\left(\frac{1}{2}\right)^i\delta_{\ell-1}$, we don't move $v_\ell^-(i)$ in the step and let $v_\ell^+(i)=v_\ell^-(i)$. Hence,

$$\measuredangle\left(v_\ell^-(i),v_\ell^+(i)\right)\leq\left(\frac{1}{2}\right)^i\delta_{\ell-1},\ \forall\ i\in\{1,...,n-1\}.$$

Notice that, for any $v\in\mathbb{S}^{n-1}(1)$, $\measuredangle\left(v,\pi_{\alpha_i}^{\ell-1}\right)\geq\delta$ means $\measuredangle\left(v,\pm e_{\alpha_i}^{\ell-1}\right)\leq\pi/2-\left(\frac{1}{2}\right)^i\delta_{\ell-1}$ for some $\pm\in\{-,+\}$. For each fixed α_i and $i\in\{1,...,n-1\}$ in (3.15), we apply the triangle inequality of the distance function on $\mathbb{S}^{n-1}(1)$ to derive

$$\measuredangle\left(v_\ell^+(n-1),v_\ell^+(i)\right)\leq\sum_{\kappa=i+1}^{n-1}\measuredangle\left(v_\ell^+(\kappa),v_\ell^-(\kappa)\right)\leq\delta_{\ell-1}\sum_{\kappa=i+1}^{n-1}\left(\frac{1}{2}\right)^\kappa.$$

Since $\left(\frac{1}{2}\right)^i\cdot\delta_{\ell-1}=\measuredangle(v_\ell^+(n-1),\pi_{\alpha_i}^{\ell-1})=\frac{\pi}{2}-\measuredangle\left(v_\ell^+(n-1),\pm e_{\alpha_i}^{\ell-1}\right)$ for some $\pm\in\{-,+\}$, we have

$$\begin{aligned}&\measuredangle\left(v_\ell^+(n-1),\pm e_{\alpha_i}^{\ell-1}\right)\leq\measuredangle\left(v_\ell^+(n-1),v_\ell^+(i)\right)+\measuredangle\left(v_\ell^+(i),\pm e_{\alpha_i}^{\ell-1}\right)\\&\leq\tfrac{\pi}{2}-\delta_{\ell-1}\cdot\left(\tfrac{1}{2}\right)^i+\delta_{\ell-1}\cdot\sum_{\kappa=i+1}^{n-1}\left(\tfrac{1}{2}\right)^\kappa=\tfrac{\pi}{2}-\left(\tfrac{1}{2}\right)^{n-1}\delta_{\ell-1}.\end{aligned}$$

Hence,

$$\measuredangle\left(v_\ell^+(n-1),\pi_{\alpha_i}^{\ell-1}\right)\geq\left(\frac{1}{2}\right)^{n-1}\delta_{\ell-1},\ \forall\ i\in\{1,...,n-1\}.\tag{3.16}$$

Similarly, for the case of $i=n$, i.e., $\alpha_i=\alpha_n$ in (3.15), we have

$$\begin{aligned}&\measuredangle\left(v_\ell^+(n-1),\pm e_{\alpha_n}^{\ell-1}\right)\leq\measuredangle\left(v_\ell^+(n-1),v_\ell\right)+\measuredangle\left(v_\ell,\pm e_{\alpha_n}^{\ell-1}\right)\\&\leq\tfrac{\pi}{2}-\delta_{\ell-1}+\delta_{\ell-1}\cdot\sum_{\kappa=1}^{n-1}\left(\tfrac{1}{2}\right)^\kappa=\tfrac{\pi}{2}-\left(\tfrac{1}{2}\right)^{n-1}\delta_{\ell-1}.\end{aligned}\tag{3.17}$$

Hence,

$$\measuredangle\left(v_\ell^+(n-1),\pi_{\alpha_n}^{\ell-1}\right)\geq\left(\frac{1}{2}\right)^{n-1}\delta_{\ell-1}.\tag{3.18}$$

Thus, from (3.16) and (3.18), we derive

$$\measuredangle\left(v_\ell^+(n-1),\pi_{\alpha_i}^{\ell-1}\right)\geq\left(\frac{1}{2}\right)^{n-1}\delta_{\ell-1},\ \forall\ i\in\{1,...,n\}\tag{3.19}$$

which is equivalent to letting $v_\ell=v_\ell^+(n-1)$ under the new coordinate basis.

We should also notice here that as we rotate the coordinate basis of $\mathbb{R}^n$ with respective to moving a point along a geodesic on the unit sphere $\mathbb{S}^{n-1}(1)$ for a geodesic distance δ, any point on $\mathbb{S}^{n-1}(1)$ can be also considered as moving on the unit sphere with respective to the rotation of coordinate basis of $\mathbb{R}^n$ for a geodesic distance no greater than δ (because the rotation is a rigid motion). Hence, under the rotation of coordinate basis of $\mathbb{R}^n$ to adjust the position of v_ℓ relative to the new coordinate basis, we have the estimates,

$$\measuredangle\left(v_b^-(i),v_b^+(i)\right)\leq\left(\frac{1}{2}\right)^i\delta_{\ell-1},\ \forall\ i\in\{1,...,n-1\}$$

for each $b \in \{1, ..., \ell - 1\}$. Thus, for each fixed $b \in \{1, ..., \ell - 1\}$, we may use the assumption in (3.13) and then follow the same computation in (3.17) to derive

$$\measuredangle(v_b^+(n-1), \pi_{\alpha_i}^{\ell-1}) \geq \left(\frac{1}{2}\right)^{n-1} \delta_{\ell-1} \tag{3.20}$$

for all $i \in \{1, ..., n\}$.

From (3.19) and (3.20), we conclude that, for all $b \in \{1, ..., \ell\}$ and all $i \in \{1, ..., n\}$,

$$\measuredangle(v_b, \pi_i^{\ell}) \geq \left(\frac{1}{2}\right)^{n-1} \delta_{\ell-1} = \delta_\ell \tag{3.21}$$

where π_i^ℓ is with respective to the new coordinate basis of $\mathbb{R}^n$ after adding the extra point v_ℓ into the set $\{v_1, ..., v_{\ell-1}\}$ and the last equality comes from the definition of δ_ℓ in (3.12).

Since there are at most Q distinct points in the support of any $q = \sum_{j=1}^{Q} [\![q_j]\!] \in \mathbf{Q}_Q(\mathbb{R}^n)$, there are at most $Q(Q-1)$ non-zero vectors $q_i - q_j$, generated from $\mathrm{spt}(q)$. Thus, we conclude (3.11) from letting $\ell = Q(Q-1)$ in (3.21) and

$$\theta_0 = \theta_0(n, Q) := \delta_{Q(Q-1)} = \left(\frac{1}{2}\right)^{(n-1)(Q^2-Q-1)} \sin^{-1}\left(\frac{1}{\sqrt{n}}\right).$$

□

Remark 1. *In fact, we can choose*

$$\theta_0 = \left(\frac{1}{2}\right)^{(n-1)(\frac{Q(Q-1)}{2}-1)} \sin^{-1}\left(\frac{1}{\sqrt{n}}\right)$$

in Proposition 2 by letting $\ell = Q(Q-1)/2$ in the final paragraph of the proof. This is because that in adjusting the position of points in $\bigcup_{i,j \in \{1,\dots,Q\}} \left\{\frac{q_i - q_j}{|q_i - q_j|} : q_i \neq q_j\right\}$ relative to the new coordinate basis, the same procedure applied to $\frac{q_i - q_j}{|q_i - q_j|}$ also works for $-\frac{q_i - q_j}{|q_i - q_j|}$.

Definition 5 (The admissible closed balls of q in $\mathbf{Q}_Q(\mathbb{R}^n)$ and the union of admissible closed balls of $\mathrm{spt}(q)$ in $\mathbb{R}^n$)**.** *Let $q = \sum_{i=1}^{I} \ell_i [\![q_i]\!]$ be any member in $\mathbf{Q}_Q(\mathbb{R}^n)$, where $q_1, ..., q_I$ are distinct points in $\mathbb{R}^n$. Denote by*

$$\mathbb{B}_\tau^n(y_0) := \{y \in \mathbb{R}^n : |y - y_0| \leq \tau\}$$

the closed ball of radius τ with center y_0 in $\mathbb{R}^n$, by

$$\mathcal{U}_\tau(q) := \bigcup_{i=1}^{I} \mathbb{B}_\tau^n(q_i)$$

the union of closed balls of radius τ in $\mathbb{R}^n$, and by

$$\mathbb{B}_\tau^{\mathbf{Q}}(q) := \{p \in \mathbf{Q}_Q(\mathbb{R}^n) : \mathcal{G}(p, q) \leq \tau\} \subset \mathbf{Q}_Q(\mathbb{R}^n)$$

the closed ball of radius τ with center q in $\mathbf{Q}_Q(\mathbb{R}^n)$. Then, $\mathcal{U}_\tau(q)$ is said to be admissible *if*

$$\Pi_\varkappa\left(\mathbb{B}^n_\tau(q_i)\right) \cap \Pi_\varkappa\left(\mathbb{B}^n_\tau(q_j)\right) = \emptyset, \ \forall\ \varkappa \in \{1, ..., n\},\ \forall\ i \neq j \in \{1, ..., I\}.$$

The closed ball $\mathbb{B}^{\mathbf{Q}}_\tau(q)$ is said to be admissible *if $\mathcal{U}_\tau(q)$ is* admissible.

From Proposition 2, if τ satisfies

$$0 < \tau < \frac{\sin\theta_0(n,Q)}{2} \cdot \inf_{i\neq j}\{|q_i - q_j|\}$$

where $\theta_0(n,Q)$ is given in Proposition 2 as $n \geq 2$ and $\theta_0(1,Q) := \pi/2$, then $\mathcal{U}_\tau(q)$ is a union of admissible closed ball of radius τ in $\mathbb{R}^n$.

We introduce the notion of *admissible* closed balls in Definition 5 for the construction of smooth vector fields in $\mathbb{R}^n$ in the range-variations of multiple-valued functions. There is a formula of "subtraction" between $q \in \mathbf{Q}_Q(\mathbb{R}^n)$ and any member in the admissible ball $\mathbb{B}^{\mathbf{Q}}_\tau(q) \subset \mathbf{Q}_Q(\mathbb{R}^n)$ of q (under a suitable choice of coordinate basis of $\mathbb{R}^n$). In other words, for any $p = \sum_{j=1}^{Q} [\![p_j]\!] \in \mathbb{B}^{\mathbf{Q}}_\tau(q)$, an admissible closed ball of q ($p_1, ..., p_Q$ are not necessarily distinct), there is a natural way to obtain an member in $\mathbf{Q}_Q(\mathbb{R}^n)$, denoted as $p \ominus q$ or $q \ominus p$, such that $\mathcal{G}\left(p \ominus q, Q[\![0]\!]\right) = \mathcal{G}\left(p, q\right)$. To explain it, observe that $p \in \mathbb{B}^{\mathbf{Q}}_\tau(q)$ implies $\mathrm{spt}(p) \subset \mathcal{U}_\tau\left(q\right)$ and $\mathrm{card}\left(\mathrm{spt}(p) \cap \mathbb{B}_\tau\left(q_i\right)\right) = \ell_i$ for each $i \in \{1, ..., I\}$. Otherwise, there exists $i \in \{1, ..., I\}$ such that $\mathrm{card}\left(\mathrm{spt}(p) \cap \mathbb{B}_\tau\left(q_i\right)\right) \lneqq \ell_i$. But this means that there is a point $p_\iota \in \mathrm{spt}(p)$ such that $|p_\iota - q_i| \gneqq \tau$, which contradicts the assumption of $p \in \mathbb{B}^{\mathbf{Q}}_\tau(q)$. Thus, we may write $p = \sum_{j=1}^{Q} [\![p_j^{\varkappa_j}]\!]$, where $\mathrm{card}\left(\{j \in \{1, ..., Q\} : \varkappa_j = i\}\right) = \ell_i$, $p_j^{\varkappa_j} \in \mathbb{B}_\tau\left(q_{\varkappa_j}\right)$, $\varkappa_j \in \{1, ..., I\}$, for each $j \in \{1, ..., Q\}$. Furthermore, for any p in an admissible closed ball of q, $\mathbb{B}^Q_\tau(q)$, we can define the "subtraction" $q \ominus p \in \mathbf{Q}_Q(\mathbb{R}^n)$ by

$$q \ominus p := \sum_{j=1}^{Q} [\![q_{\varkappa_j} - p_j^{\varkappa_j}]\!]. \tag{3.22}$$

It is obvious that

$$\mathcal{G}\left(q \ominus p, Q[\![0]\!]\right) = \mathcal{G}\left(p, q\right), \forall p \in \mathbb{B}^Q_\tau(q). \tag{3.23}$$

Let $q = \sum_{i=1}^{I} \ell_i [\![q_i]\!] \in \mathbf{Q}_Q(\mathbb{R}^n)$. Recall from (2.2) that we can choose the map $\xi(\Pi_\alpha, \cdot) : \mathbf{Q}_Q(\mathbb{R}^n) \to \mathbb{R}^Q$ for all $\alpha \in \{1, ..., n\}$. Then, for each fixed $\alpha \in \{1, ..., n\}$, we may write

$$\xi(\Pi_\alpha, q) = \left\{\left(\Pi_\alpha\left(q_{\varsigma_\alpha(1)}\right), \cdots, \Pi_\alpha\left(q_{\varsigma_\alpha(I)}\right)\right) : \Pi_\alpha\left(q_{\varsigma_\alpha(1)}\right) \leq \cdots \leq \Pi_\alpha\left(q_{\varsigma_\alpha(I)}\right)\right\} \subset \mathbb{R}^Q$$

and

$$\xi(\Pi_\alpha, p) = \left\{\left(\Pi_\alpha\left(p^{\varsigma_\alpha(1)}_{\omega_\alpha(1)}\right), \cdots, \Pi_\alpha\left(p^{\varsigma_\alpha(I)}_{\omega_\alpha(Q)}\right)\right) : \Pi_\alpha\left(p^{\varsigma_\alpha(1)}_{\omega_\alpha(1)}\right) \leq \cdots \leq \Pi_\alpha\left(p^{\varsigma_\alpha(I)}_{\omega_\alpha(Q)}\right)\right\}$$

for some permutation $\varsigma_\alpha : \{1, ..., I\} \to \{1, ..., I\}$ and $\omega_\alpha : \{1, ..., Q\} \to \{1, ..., Q\}$. Notice that, as a member $p \in \mathbf{Q}_Q(\mathbb{R}^n)$ is contained in an admissible closed ball of q, the two types of "subtraction", $q \ominus p$ and $\xi(\Pi_\alpha, q) - \xi(\Pi_\alpha, p) \in \mathbb{R}^Q$ are consistent, $\forall\ \alpha \in \{1, ..., n\}$. In other words, if $\mathbb{B}^{\mathbf{Q}}_\tau(q) \subset \mathbf{Q}_Q(\mathbb{R}^n)$ is an admissible closed ball of q and $p \in \mathbb{B}^{\mathbf{Q}}_\tau(q)$, then we may define $p(s) := s \cdot q + (1 - s) \cdot p$ and $s \in \mathbb{R}$ by

$$p(s) = \sum_{j=1}^{Q} [\![s \cdot \left(q_{\varkappa_j} - p_j^{\varkappa_j}\right) + (1 - s) \cdot p_j^{\varkappa_j}]\!] \tag{3.24}$$

and $p(s)$ satisfies the property

$$\xi\left(\Pi_\alpha, p(s)\right) = s \cdot \xi(\Pi_\alpha, p(1)) + (1 - s) \cdot \xi(\Pi_\alpha, p(0)).$$

Hence,

$$\boldsymbol{\xi}_0\left(p(s)\right) = s \cdot \boldsymbol{\xi}_0(q) + (1 - s) \cdot \boldsymbol{\xi}_0(p), \tag{3.25}$$

for all $p \in \mathbb{B}^{\mathbf{Q}}_\tau(q)$ (an admissible closed ball of q). From (3.22), (3.23) and taking $\frac{d}{ds}$ in (3.25), we conclude that

$$|\boldsymbol{\xi}_0(q \ominus p)|^2 = |\boldsymbol{\xi}_0(q) - \boldsymbol{\xi}_0(p)|^2 = \left[\mathcal{G}\left(q \ominus p, Q[\![0]\!]\right)\right]^2 = \left[\mathcal{G}\left(p, q\right)\right]^2, \tag{3.26}$$

for any p in an admissible closed ball $\mathbb{B}^{\mathbf{Q}}_\tau(q)$.

Proposition 3. *For any $q = \sum_{i=1}^{I} \ell_i [\![q_i]\!] \in \mathbf{Q}_Q(\mathbb{R}^n)$, there correspond a non-negative integer $L \in \{0, 1, ..., Q-1\}$, a sequence of members $q^{(0)} = q, q^{(1)}, ..., q^{(L)} \in \mathbf{Q}_Q(\mathbb{R}^n)$, where $q^{(k)} = \sum_{i=1}^{I_k} \ell_i^k [\![q_i^k]\!]$, a constant $C_0 = C_0(n, Q) > 1$, and a sequence of numbers,*

$$0 =: \rho_0 < \sigma_0 < \rho_1 < \sigma_1 < \rho_2 < \cdots < \sigma_{L-1} < \rho_L < \sigma_L := +\infty \tag{3.27}$$

such that the following statements hold.

(a) *If $card\,(spt(q)) = 1$, then $L = 0$, $\rho_0 = 0$ and $\sigma_0 = +\infty$.*

(b) *If $card\,(spt(q)) \geq 2$, then $L \geq 1$,*

$$spt\left(q^{(k-1)}\right) \supsetneq spt\left(q^{(k)}\right) \tag{3.28}$$

for each $k \in \{1, ..., L\}$ and the set $spt\left(q^{(L)}\right)$ is consisted of a single point.

(c) *For each $k \in \{0, 1, ..., L-1\}$ and $L \geq 1$,*

$$10Q \cdot \rho_k \leq \sigma_k \tag{3.29}$$

where

$$\sigma_k := \frac{\sin\theta_0}{4} \cdot \inf_{i \neq j} \left\{ |q_i^k - q_j^k|) \right\} \tag{3.30}$$

$\theta_0(n, Q) \in (0, \pi/4)$ is given in Proposition 2 as $n \geq 2$ and $\theta_0(n, Q) = \pi/2$ as $n = 1$.

(d) *For each $k \in \{1, ..., L\}$ and $L \geq 1$,*

$$\sigma_{k-1} < \rho_k \leq C_0(n, Q) \cdot \sigma_{k-1}. \tag{3.31}$$

(e) *For each* $k \in \{1, ..., L\}$ *and* $L \geq 1$, *if* $\sigma_{k-1} < \rho_k$, *then*

$$\mathbb{B}^{\mathbf{Q}}_{\sigma_{k-1}}\left(q^{(k-1)}\right) \subset \mathbb{B}^{\mathbf{Q}}_{\rho_k}\left(q^{(k)}\right). \tag{3.32}$$

(f) *For each* $k \in \{1, ..., L\}$ *and* $L \geq 1$,

$$\mathcal{G}\left(q^{(0)}, q^{(k)}\right) \leq \rho_1 + \cdots + \rho_k < (Q-1)\cdot\rho_k. \tag{3.33}$$

Proof. For a given $q = \sum_{i=1}^{I} \ell_i [\![q_i]\!] \in \mathbf{Q}_Q(\mathbb{R}^n)$, denote by $q^{(0)} = q$ and $q^{(0)} = \sum_{i=1}^{I_0} \ell_i^0 [\![q_i^0]\!]$. Without loss of generality, we may assume that $\operatorname{card}(\operatorname{spt}(q)) \geq 2$. Then we follow the so-called *standard modification procedure for members of* $\mathbf{Q}_Q(\mathbb{R}^n)$ in [1, 2.9] to find $q^{(1)}$ by choosing sufficiently large constant K in [1, 2.9] (see (3.34) below). We successively apply this procedure to obtain $q^{(k+1)}$ from $q^{(k)}$ until $\operatorname{card}\left(\operatorname{spt}(q^L)\right) = 1$ for some $L \in \mathbb{Z}_+$. Such positive integer $L \leq Q-1$ fulfilling $\operatorname{card}\left(\operatorname{spt}(q^{(L)})\right) = 1$ exists, because we will show that $\operatorname{card}\left(\operatorname{spt}(q^{(k)})\right) \gneqq \operatorname{card}\left(\operatorname{spt}(q^{(k+1)})\right)$ in the procedure.

Note that the assertion in Proposition is obviously fulfilled as $k = 0$. Hence, by induction argument, we suppose that the assertion is true for $q^{(0)}, ..., q^{(k)} \in \mathbf{Q}_Q(\mathbb{R}^n)$, where $k \geq 1$ and $\operatorname{card}\left(\operatorname{spt}(q^{(k)})\right) \geq 2$ (otherwise, the proof is finished). Now, we show how to find $q^{(k+1)}$ from $q^{(k)}$. We first define some numbers for the construction of the sequence in (3.27) as follows. Let

$$K = K(n, Q) = \frac{20Q}{\sin\theta_0(n, Q)} \tag{3.34}$$

and

$$s_0 = \sigma_k \tag{3.35}$$

where σ_k is given in (3.30). The choice of K in (3.34) assures that the quotient σ_k/ρ_k is sufficient large, see (3.29). Besides, for a constructed $q^{(0)}, ..., q^{(k)}$, a sufficiently large K and a properly chosen σ_k assure that $\operatorname{card}\left(\operatorname{spt}(q^{(k)})\right) \gneqq \operatorname{card}\left(\operatorname{spt}(q^{(k+1)})\right)$, in the following procedure. Let

$$\begin{cases} t_1 = 0, \\ d_\varkappa = (Q-1)t_\varkappa, \text{ for each } \varkappa = 1, 2, ..., \\ s_\varkappa = (Q-1)d_\varkappa + s_{\varkappa-1}, \text{ for each } \varkappa = 1, 2, ..., \\ t_{\varkappa+1} = 2K\cdot s_\varkappa, \text{ for each } \varkappa = 1, 2, ... \end{cases} \tag{3.36}$$

By a direct computation from (3.36) and restricting $\varkappa \in \{1, ..., Q\}$, we could derive

$$\begin{cases} s_0 = s_1, \\ s_\varkappa = [1 + 2K(Q-1)^2]^{\varkappa-1}s_0 \leq [1 + 2K(Q-1)^2]^{Q-1}s_0, \\ d_\varkappa = 2K(Q-1)[1 + 2K(Q-1)^2]^{\varkappa-2}\cdot s_0. \end{cases} \tag{3.37}$$

For each fixed $\varkappa \in \{1, ..., Q\}$, define a partitioning of $\operatorname{spt}(q^{(k)}) = \{q_1^k, ..., q_{I_k}^k\}$ into equivalence classes by saying that $q_i^k \sim q_j^k$ if there exists a sequence $\{q_{i_1}^k, ..., q_{i_A}^k\} \subset \operatorname{spt}(q^{(k)})$ such that $q_i^k = q_{i_1}^k$, $q_j^k = q_{i_A}^k$ and $|q_{i_\alpha}^k - q_{i_{\alpha+1}}^k| \leq t_k$ for each $\alpha =$

$1, 2, ..., A-1$. Denote by $P(\varkappa, 1), ..., P(\varkappa, N_k(\varkappa)) \subset \{q_1^k, ..., q_{I_k}^k\}$ a list of the distinct equivalence classes, where $N_k(\varkappa)$ represents the number of distinct equivalence classes at the $\varkappa$-th stage of this partitioning procedure. It is easy to see that

$$\operatorname{diam}(P(\varkappa, i)) \leq d_\varkappa, \ \forall \ \varkappa \text{ and } i \tag{3.38}$$

and $Q \geq N_k(1) \geq N_k(2) \geq \cdots \geq 1$. Denote by $\varkappa_0$ the smallest positive integer among the integers $\varkappa$ so that $N_k(\varkappa) = N_k(\varkappa + 1)$. It is clear that $\varkappa_0 \leq Q$. Furthermore, let

$$\rho_{k+1} := s_{\varkappa_0} \tag{3.39}$$

and, for each $i \in \{1, ..., N_k(\varkappa_0)\}$, choose some $q_i^{k+1} \in P(\varkappa_0, i)$ and let $\ell_i^{k+1} = \operatorname{card}(P(\varkappa_0, i))$.

From (3.38), it is easy to check that for each fixed integer $k \geq 0$,

$$\begin{aligned} \left[\mathcal{G}\left(q^{(k)}, q^{(k+1)}\right)\right]^2 &= \left[\mathcal{G}\left(\sum_{i=1}^{I_k} \ell_i^k [\![q_i^k]\!], \sum_{j=1}^{I_{k+1}} \ell_j^{k+1} [\![q_j^{k+1}]\!]\right)\right]^2 \\ \leq (Q-1) \cdot \left[\sup_i \{\operatorname{diam}(P(\varkappa_0, i))\}\right]^2 &\leq (Q-1) \cdot d_{\varkappa_0}^2. \end{aligned} \tag{3.40}$$

Moreover, for any $z \in \mathbf{Q}_Q(\mathbb{R}^n)$ satisfying $\mathcal{G}\left(z, q^{(k)}\right) \leq s_0$, we may apply the triangle inequality of $\mathcal{G}(\cdot, \cdot)$ in $\mathbf{Q}_Q(\mathbb{R}^n)$, and (3.37), (3.40) to derive

$$\begin{aligned} &\mathcal{G}\left(z, q^{(k+1)}\right) \leq \mathcal{G}\left(z, q^{(k)}\right) + \mathcal{G}\left(q^{(k)}, q^{(k+1)}\right) \\ &\leq s_0 + (Q-1)^{1/2} \cdot d_{\varkappa_0} \\ &= \left(1 + 2K(Q-1)^{3/2}[1 + 2K(Q-1)^2]^{\varkappa_0 - 2}\right) \cdot s_0 \\ &\leq [1 + 2K(Q-1)^2]^{\varkappa_0 - 1} \cdot s_0 = s_{\varkappa_0} \end{aligned} \tag{3.41}$$

where the last equality comes from (3.37). Hence, from (3.39) and (3.35), we have

$$\mathcal{G}\left(z, q^{(k+1)}\right) \leq \rho_{k+1}$$

if z satisfies $\mathcal{G}\left(z, q^{(k)}\right) \leq \sigma_k$. Note that, due to the choice of K in (3.34) and by letting $s_0 = \sigma_k$ in (3.35), we derive from (3.36),

$$t_2 = 10Q \cdot \inf_{i \neq j} \left\{|q_i^k - q_j^k|\right\} > \inf_{i \neq j} \left\{|q_i^k - q_j^k|\right\}.$$

This implies that $\varkappa_0 \geq 2$ and therefore

$$\operatorname{card}\left(\operatorname{spt}(q^{(k)})\right) \gneqq \operatorname{card}\left(\operatorname{spt}(q^{(k+1)})\right). \tag{3.42}$$

Thus, from (3.39) and (3.30), we have $\sigma_k \lneqq \rho_{k+1}$. Besides, from (3.39), (3.35) and (3.37), we have

$$\frac{\rho_{k+1}}{\sigma_k} = [1 + 2K(Q-1)^2]^{\varkappa_0 - 1} \leq [1 + 2K(Q-1)^2]^{Q-1}.$$

Now we let

$$C_0(n, Q) := [1 + 2K(Q-1)^2]^{Q-1}.$$

On the other hand, the choice of $\varkappa_0$ implies that,

$$|z_i - z_j| > t_{\varkappa_0 + 1} = 2K \cdot s_{\varkappa_0} = 2K \cdot \rho_{k+1} \tag{3.43}$$

for any $z_i \in P(\varkappa_0, i)$, $z_j \in P(\varkappa_0, j)$ and any distinct equivalence classes $P(\varkappa_0, i)$, $P(\varkappa_0, j)$. Hence, from (3.30) and (3.34), (3.43) implies

$$\sigma_{k+1} \geq \frac{\sin\theta_0}{4} \cdot t_{\varkappa_0+1} = 10Q \cdot \rho_{k+1}.$$

Notice that, due to the strictly decreasing of cardinality in (3.42), we only proceed this procedure in constructing $q^{(k+1)}$ from $q^{(k)}$ for at most $(Q-1)$ many times (until spt $\left(q^{(L)}\right)$ is consisted of only one point). Hence, $L \in \{1, ..., Q-1\}$.

The proof of each statement from (a) to (e) follows from the argument above for all $k \in \{1, ..., L\}$ inductively. The proof of (f) follows from applying (e) and the triangle inequality of the metric space $(\mathbf{Q}_Q(\mathbb{R}^n), \mathcal{G}(\cdot,\cdot))$.

□

Definition 6 (The nested admissible closed balls of $q \in \mathbf{Q}_Q(\mathbb{R}^n)$)**.** *We follow the notations in Proposition 3. Let $\left\{\mathbb{B}^{\mathbf{Q}}_{\tau_k}(q^{(k)})\right\}_{k=0}^{L}$ fulfill $\tau_k \in [\rho_k, \sigma_k]$ for each $k \in \{0, 1, ..., L\}$, as stated in Proposition 3. Then, $\left\{\mathbb{B}^{\mathbf{Q}}_{\tau_k}(q^{(k)})\right\}_{k=0}^{L}$ is said to be a sequence of nested admissible closed balls of q.*

The nested admissible closed balls of any member in $\mathbf{Q}_Q(\mathbb{R}^n)$ will be useful for establishing the "global" monotonicity formula in the proof of Lemma 3.

Lemma 3 (Key Lemma)**.** *Assume $\Omega \subset \mathbb{R}^2$ is an open set, $\mathbb{U}_r(w) \subset\subset \Omega$ is an open ball of radius $r > 0$ with the center w and $f \in \mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ is weakly stationary-harmonic. Let $h : \Omega \to \mathbb{R}^2$ be the harmonic function induced from the Hopf-differential of $\boldsymbol{\xi}_0 \circ f$ as constructed in Proposition 1. Suppose $A \subset \Omega$ is a set of Lebesgue points of $|\nabla(\boldsymbol{\xi}_0 \circ f)|^2$ and $w^* \in A \cap \mathbb{U}_r(w)$ fulfills $\inf\limits_{x\in\partial\mathbb{U}_r(w)} \mathcal{G}(f(x), f(w^*)) > 0$. Then,*

$$\inf_{x\in\partial\mathbb{U}_r(w)} \mathcal{G}(f(x), f(w^*)) \leq \sqrt{\frac{Dir\,(f; \mathbb{U}_r(w)) + Dir\,(h; \mathbb{U}_r(w))}{2\pi \cdot \delta(n,Q)}} \tag{3.44}$$

for some constant $\delta(n, Q) > 0$.

Remark 2. *The set $A \subset \Omega$, defined as the Lebesgue point of $|\nabla(\boldsymbol{\xi}_0 \circ f)|^2$ in Lemma 3, is independent of the choice of $\boldsymbol{\xi}_0$ (since $|\nabla(\boldsymbol{\xi}_0 \circ f)|$ is invariant with respective to the choice of $\boldsymbol{\xi}_0$). Besides, $\mathcal{L}^2(\Omega \setminus A) = 0$, because $\boldsymbol{\xi}_0 \circ f \in W^{1,2}(\Omega, \mathbb{R}^N)$. Recall from Proposition 1 that $G = (\boldsymbol{\xi}_0 \circ f, h)$ and*

$$|\nabla G|^2 = |\nabla(\boldsymbol{\xi}_0 \circ f)|^2 + |\nabla h|^2 = |\nabla(\boldsymbol{\xi}_0 \circ f)|^2 + \frac{|\varphi|^2}{8} + 2 \geq 2. \tag{3.45}$$

From (3.45) and the definition of "good" points in Definition 1, $A \subset \Omega$ is a set of "good" points of G.

For our convenience, let $F = \boldsymbol{\xi}_0 \circ f \in \mathbb{R}^{nQ}$, $G = (F, h) \in \mathbb{R}^{nQ+2}$, and

$$d^*(x) := \sqrt{\mathcal{G}(f(w^*), f(x))^2 + |h(w^*) - h(x)|^2}. \tag{3.46}$$

Hence, (3.44) is equivalent to

$$\inf_{x\in\partial\mathbb{U}_r(w)} d^*(x) \leq \sqrt{\frac{Dir\left(G;\mathbb{U}_r(w)\right)}{2\pi\cdot\delta(n,Q)}}. \tag{3.47}$$

Proof. Let $\mathbb{U}_{R_0}(w)\subset\Omega$, where $w\in\Omega$ and $R_0>0$. Without loss of generality, assume that f is not identically the constant $f(w^*)$ on $\partial B_r(w)$. Hence, we let

$$\tau_* := \inf_{x\in\partial\mathbb{U}_r(w)} \mathcal{G}(f(w^*), f(x)) > 0.$$

We follow the notations in Proposition 3 and let $f(w^*)=q^{(0)}$, where $w^*\in A\cap\mathbb{U}_r(w)$. From Proposition 2, we may also choose a suitable coordinate basis of $\mathbb{R}^n$ for the construction of admissible closed balls of w^*. Since the Dirichlet integral is invariant under the change of Cartesian coordinates of $\mathbb{R}^n$, for our convenience, we still denote by f the multiple-valued function after composed with the change of coordinates of $\mathbb{R}^n$. Then, from Proposition 3, there correspond two sequences of non-negative numbers $\{\rho_k\}_{k=0}^L$, $\{\sigma_k\}_{k=0}^L$ and a sequence of members $\{q^{(k)}\}_{k=0}^L\subset\mathbf{Q}_Q(\mathbb{R}^n)$. For a given positive number τ_*, let $k_0\in\{0,1,...,L\}$ be the integer fulfilling one of the following conditions,

$$\begin{cases} 10Q\cdot\rho_{k_0}\lneqq\tau_*\leq 10Q\cdot\rho_{k_0+1},\ k_0\in\{0,1,...,L-1\},\\ 10Q\cdot\rho_L\lneqq\tau_*<+\infty,\ k_0=L.\end{cases} \tag{3.48}$$

Let

$$d_k^*(x) := \sqrt{\mathcal{G}(q^{(k)},f(x))^2+|h(w^*)-h(x)|^2} \tag{3.49}$$

and

$$\Omega_k^*(\rho) := \{x\in\Omega : d_k^*(x)<\rho\} \tag{3.50}$$

for each $k\in\{0,1,...,L\}$. For any x satisfying one of the following conditions

$$\begin{cases} d_k^*(x)\leq\sigma_k,\ \text{if } k\in\{0,...,k_0-1\},\\ d_{k_0}^*(x)\leq\frac{2}{5}\min\{\tau_*,\sigma_{k_0}\},\end{cases} \tag{3.51}$$

one could verify that

$$\begin{aligned} d_0^*(x) &:= \sqrt{\mathcal{G}(q^{(0)},f(x))^2+|h(w^*)-h(x)|^2} \leq \mathcal{G}\left(q^{(0)},f(x)\right)+|h(w^*)-h(x)| \\ &\leq \mathcal{G}\left(q^{(0)},q^{(k)}\right)+\mathcal{G}\left(q^{(k)},f(x)\right)+|h(x)-h(w^*)| \leq \mathcal{G}\left(q^{(0)},q^{(k)}\right)+\sqrt{2}\cdot d_k^*(x).\end{aligned} \tag{3.52}$$

As $k\in\{0,...,k_0-1\}$, we apply (3.51), (3.27), (3.33) and (3.48) to derive

$$\begin{aligned} &\text{R.H.S. of (3.52)} \lneqq \mathcal{G}\left(q^{(0)},q^{(k)}\right)+\sqrt{2}\cdot\sigma_k \\ &< (Q-1)\cdot\rho_k+\sqrt{2}\cdot\sigma_k < (1+\sqrt{2})\cdot\sigma_k < (1+\sqrt{2})\cdot\rho_{k_0} < \tau_*.\end{aligned}$$

As $k=k_0$, we apply (3.51) to derive

$$\begin{aligned}
&\text{R.H.S. of (3.52)} \lneqq \mathcal{G}\left(q^{(0)}, q^{(k_0)}\right) + \tfrac{2\sqrt{2}}{5}\cdot\min\{\tau_*, \sigma_{k_0}\} \\
&= \min\{\tau_*, \sigma_{k_0}\} - \left(\tfrac{5-2\sqrt{2}}{5}\cdot\min\{\tau_*, \sigma_{k_0}\} - \mathcal{G}\left(q^{(0)}, q^{(k_0)}\right)\right) \\
&\lneqq \min\{\tau_*, \sigma_{k_0}\},
\end{aligned}$$

where the last inequality comes from applying (3.48), (3.29) and (3.33). Therefore,

$$\begin{cases} \rho \in (0, \sigma_k)\,, \text{ as } k = 0, ..., k_0 - 1, \\ \rho \in \left(0, \tfrac{2}{5}\min\{\tau_*, \sigma_{k_0}\}\right)\,, \text{ as } k = k_0. \end{cases} \Longrightarrow \Omega_k^*(\rho) \subset\subset \mathbb{U}_r(w). \tag{3.53}$$

In the rest of this article, for a given $k \in \{0, ..., k_0\}$, we always let ρ fulfill the condition in (3.53).

Step 1: Constructing "admissible" range-variations.

Define the smooth vector field $\Gamma_k : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Gamma_k(y) := \chi(|q_i^k - y|)\cdot\left(q_i^k - y\right)$$

where $\chi : \mathbb{R} \to [0,1]$ is a smooth function satisfying $\chi(s) = 1$ as $s \le \frac{2}{5}\sigma_k$, $\chi(s) = 0$ as $s \ge \frac{3}{5}\sigma_k$. Hence, $\mathrm{Lip}(\Gamma_k) \le \frac{5}{\sigma_k}$. The $\mathbf{Q}_Q(\mathbb{R}^n)$-valued function, induced from the smooth vector field $\Gamma_k : \mathbb{R}^n \to \mathbb{R}^n$, can be written as

$$(\Gamma_k)_\#\,(f(x)) := \sum_{i=1}^{Q} [\![\Gamma_k \circ f_i(x)]\!]. \tag{3.54}$$

Note, the definition of $\Omega_k^*(\rho)$ in (3.50) and the definition of d_k^* in (3.49) give us the condition

$$\rho \le \frac{2}{5}\sigma_k \Longrightarrow f\left(\Omega_k^*(\rho)\right) \subset \mathbb{B}^{\mathbf{Q}}_{\frac{2}{5}\sigma_k}(q^{(k)}). \tag{3.55}$$

Let $\lambda \in C^\infty(\mathbb{R}, [0,1])$ be given by satisfying $\lambda'(s) \ge 0$ and

$$\lambda(s) = \begin{cases} 0\,, \text{ if } s \le 0, \\ 1\,, \text{ if } s \ge \varepsilon, \end{cases} \tag{3.56}$$

for some $\varepsilon > 0$. From (3.1), we let the range-variation of f be

$$f^t(x) := \sum_{i=1}^{Q} [\![f_i(x) + t\cdot\lambda(\rho - d_k^*(x))\cdot\Gamma_k(f_i(x))]\!] \tag{3.57}$$

where $t \in (-\epsilon, \epsilon)$ and $\epsilon > 0$ is a sufficiently small number. It remains to show that the range-variation of f in (3.57) is admissible. In other words, for each fixed t, we should prove that $\{f^t\}$ belongs to the class of Sobolev space $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ and $\{f^t\}$ could be approximated by a sequence of multiple-valued functions generated from a smooth perturbation of f (as defined in Definition 2). In the following, we want to show the local fine-property of Λ_k and f^t.

We first prove that $\Lambda_k : \Omega \to \mathbb{R}$ belongs to the class of Sobolev space $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ by the difference quotient method. Observe that

$$\frac{|\Lambda_k(x_2)-\Lambda_k(x_1)|}{|x_2-x_1|} = \frac{|\Lambda_k(\rho-d_k^*(x_2))-\Lambda_k(\rho-d_k^*(x_1))|}{|\left(\rho-d_k^*(x_2)\right)-\left(\rho-d_k^*(x_1)\right)|} \cdot \frac{|d_k^*(x_2)-d_k^*(x_1)|}{|x_2-x_1|}$$

$$\leq \mathrm{Lip}(\lambda) \cdot \frac{|(\mathcal{G}(q^{(k)},f(x_2))^2-\mathcal{G}(q^{(k)},f(x_1))^2)|+||h(w^*)-h(x_2)|^2-|h(w^*)-h(x_1)|^2|}{|x_2-x_1|\cdot|d_k^*(x_2)+d_k^*(x_1)|}$$

$$\leq \mathrm{Lip}(\lambda) \cdot \left(\frac{|\mathcal{G}(q^{(k)},f(x_2))-\mathcal{G}(q^{(k)},f(x_1))|}{|x_2-x_1|} + \frac{||h(w^*)-h(x_2)|-|h(w^*)-h(x_1)||}{|x_2-x_1|}\right)$$

$$\leq \mathrm{Lip}(\lambda) \cdot \left(\frac{\mathcal{G}(f(x_2),f(x_1))}{|x_2-x_1|} + \frac{|h(x_2)-h(x_1)|}{|x_2-x_1|}\right)$$

$$\leq \mathrm{Lip}(\lambda) \cdot \left(\mathrm{Lip}(\boldsymbol{\xi}^{-1}) \cdot \frac{|\boldsymbol{\xi}\circ f(x_2)-\boldsymbol{\xi}\circ f(x_1)|}{|x_2-x_1|} + \frac{|h(x_2)-h(x_1)|}{|x_2-x_1|}\right)$$

where the third inequality comes from applying triangle inequality of the metric $\mathcal{G}(\cdot,\cdot)$. Because $\boldsymbol{\xi} \circ f \in W^{1,2}(\Omega)$ and h is a smooth harmonic function, we derive the uniform bound of the difference quotient $\frac{|\Lambda_k(x_2)-\Lambda_k(x_1)|}{|x_2-x_1|}$ in $L^2(\Omega)$-topology for any distinct x_2, x_1 in Ω. Hence, $\Lambda_k \in W^{1,2}(\Omega)$. The restriction of ρ in (3.53) and the condition of $\Lambda_k \in [0,1]$ imply that $\Lambda_k \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Now we want to prove that f^t belongs to the class of Sobolev space $\mathcal{Y}_2(\Omega, \mathbf{Q}_Q(\mathbb{R}^n))$ for each fixed t. Without loss of generality, we only need to show the case when we let $\Omega = \mathbb{U}_r(w)$ because $f^t(x) = f(x)$ for all t and $x \in \Omega \setminus \mathbb{U}_r(w)$. We re-write the equation (3.57) as

$$f^t(x) = \sum_{i=1}^{Q} [\![f_i^t(x)]\!] = \sum_{i=1}^{Q} [\![f_i(x) + t \cdot \Lambda_k(x) \cdot \Gamma_k(f_i(x))]\!]. \tag{3.58}$$

For fixed $x_1, x_2 \in \mathbb{U}_r(w)$, denote by $\sigma_{x_1,x_2} : \{1,2,...,Q\} \to \{1,2,...,Q\}$ the permutation fulfilling

$$\mathcal{G}\left(f(x_2), f(x_1)\right) = \sqrt{\sum_{\ell=1}^{Q} \left| f_{\sigma_{x_1,x_2}(\ell)}(x_2) - f_\ell(x_1) \right|^2}. \tag{3.59}$$

Then, for fixed x_1, x_2, t and α,

(3.60)

$$\left|\xi_\alpha \circ f^t(x_2) - \xi_\alpha \circ f^t(x_1)\right|^2 = \left[\mathcal{G}\left((\Pi_\alpha)_\#(f^t(x_2)), (\Pi_\alpha)_\#(f^t(x_1))\right)\right]^2$$

$$= \inf_{\sigma \in \mathcal{P}_Q} \left(\sum_{\ell=1}^{Q} \left| \Pi_\alpha \left(f_{\sigma(\ell)}^t(x_2) \right) - \Pi_\alpha \left(f_\ell^t(x_1) \right) \right|^2 \right)$$

$$\leq \sum_{\ell=1}^{Q} \left| \Pi_\alpha \left(f_{\sigma_{x_1,x_2}(\ell)}^t(x_2) \right) - \Pi_\alpha \left(f_\ell^t(x_1) \right) \right|^2 = \sum_{\ell=1}^{Q} \left| \Pi_\alpha \left(f_{\sigma_{x_1,x_2}(\ell)}^t(x_2) - f_\ell^t(x_1) \right) \right|^2$$

where the first equality comes from (2.3) and $\mathcal{P}_Q$ denotes the permutation group of $\{1, 2, ..., Q\}$. By applying the inequality, $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, we have

$$\begin{aligned}
&\sum_{\ell=1}^{Q} \left|\Pi_\alpha \left(f^t_{\sigma_{x_1,x_2}(\ell)}(x_2) - f^t_\ell(x_1)\right)\right|^2 \\
&\le \sum_{\ell=1}^{Q} 3 \cdot \left|\Pi_\alpha \left(f_{\sigma_{x_1,x_2}(\ell)}(x_2) - f_\ell(x_1)\right)\right|^2 \\
&+ \sum_{\ell=1}^{Q} 3t^2 \cdot (\Lambda_k(x_2) - \Lambda_k(x_1))^2 \cdot \left|\Pi_\alpha \left(\Gamma_k \circ f_{\sigma_{x_1,x_2}(\ell)}(x_2)\right)\right|^2 \\
&+ \sum_{\ell=1}^{Q} 3t^2 \cdot (\Lambda_k(x_1))^2 \cdot \left|\Pi_\alpha \left(\Gamma_k \circ f_{\sigma_{x_1,x_2}(\ell)}(x_2) - \Gamma_k \circ f_\ell(x_1)\right)\right|^2 .
\end{aligned} \tag{3.61}$$

Hence, from (3.60), (3.61), (3.59) and taking the sum $\sum_{\alpha=1}^{n}$, we have

$$\begin{aligned}
&\left|\boldsymbol{\xi}_0 \circ f^t(x_2) - \boldsymbol{\xi}_0 \circ f^t(x_1)\right|^2 \\
&\le \sum_{\ell=1}^{Q} 3 \cdot \left|f_{\sigma_{x_1,x_2}(\ell)}(x_2) - f_\ell(x_1)\right|^2 \\
&+ \sum_{\ell=1}^{Q} 3t^2 \cdot (\Lambda_k(x_2) - \Lambda_k(x_1))^2 \cdot \left|\Gamma_k \circ f_{\sigma_{x_1,x_2}(\ell)}(x_2)\right|^2 \\
&+ \sum_{\ell=1}^{Q} 3t^2 \cdot (\Lambda_k(x_1))^2 \cdot \left|\Gamma_k \circ f_{\sigma_{x_1,x_2}(\ell)}(x_2) - \Gamma_k \circ f_\ell(x_1)\right|^2 \\
&\le 3 \cdot \left[\mathcal{G}\left(f(x_2), f(x_1)\right)\right]^2 \\
&+ \sum_{\ell=1}^{Q} 3t^2 \cdot (\Lambda_k(x_2) - \Lambda_k(x_1))^2 \cdot \left|\Gamma_k \circ f_{\sigma_{x_1,x_2}(\ell)}(x_2)\right|^2 \\
&+ 3t^2 \cdot (\Lambda_k(x_1))^2 \cdot |\mathrm{Lip}(\Gamma_k)|^2 \cdot \left[\mathcal{G}\left(f(x_2), f(x_1)\right)\right]^2 .
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\left|\boldsymbol{\xi}_0 \circ f^t(x_2) - \boldsymbol{\xi}_0 \circ f^t(x_1)\right|^2}{|x_2 - x_1|^2} \\
&\le \frac{3[\mathcal{G}(f(x_2), f(x_1))]^2}{|x_2-x_1|^2} + \frac{3t^2(\Lambda_k(x_2)-\Lambda_k(x_1))^2}{|x_2-x_1|^2} Q \left\|\Gamma_k\right\|^2_{L^\infty} + 3t^2[\mathrm{Lip}(\Gamma_k)]^2 \frac{[\mathcal{G}(f(x_2), f(x_1))]^2}{|x_2-x_1|^2} \\
&\le 3\left(1 + t^2[\mathrm{Lip}(\Gamma_k)]^2\right)[\mathrm{Lip}(\boldsymbol{\xi}^{-1})]^2 \frac{|\boldsymbol{\xi}\circ f(x_2) - \boldsymbol{\xi}\circ f(x_1)|^2}{|x_2-x_1|^2} + \frac{3t^2(\Lambda_k(x_2)-\Lambda_k(x_1))^2}{|x_2-x_1|^2} Q \left\|\Gamma_k\right\|^2_{L^\infty} ,
\end{aligned}$$

where $\left\|\Gamma_k\right\|_{L^\infty} \le 2\sigma_k/5$ and $\mathrm{Lip}(\Gamma_k) \le 5/\sigma_k$. Since k is fixed and both $\boldsymbol{\xi} \circ f$ and Λ_k belong to the class of Sobolev spaces $W^{1,2}$, we conclude that $\frac{\left|\xi_\alpha \circ f^t(x_2) - \xi_\alpha \circ f^t(x_1)\right|}{|x_2-x_1|}$ is uniformly bounded in L^2 for all distinct x_1, x_2 in $\mathbb{U}_r(w)$ and fixed t, $\alpha \in \{1, ..., n\}$.

By the difference quotient method, we conclude that f^t belongs to the Sobolev space $\mathcal{Y}_2\left(\mathbb{U}_r(w), \mathbf{Q}_Q(\mathbb{R}^n)\right)$ for each fixed t.

We may also follow the same argument above to show that the induced multiple-valued function, $\Big(\Lambda_k(x)\cdot(\Gamma_k)_{\#}\left(f(x)\right)\Big) = \sum_{i=1}^{Q}[\![\Lambda_k(x)\cdot\Gamma_k(f_i(x))]\!]$, belongs to the Sobolev space $\mathcal{Y}_2\left(\Omega, \mathbf{Q}_Q(\mathbb{R}^n)\right)$ and we write

$$\text{ap } A_x\Big(\Lambda_k(x)\cdot(\Gamma_k)_{\#}\left(f(x)\right)\Big) = \sum_{i=1}^{Q}[\![\text{ap } D_x\left(\Lambda_k(x)\cdot\Gamma_k(f_i(x))\right)]\!] \text{ for } \mathcal{L}^2 \text{ a.e. } x\in\Omega.$$

Below we show that $\frac{d}{dt}_{|t=0}Dir(f^t;\Omega)=0$ by comparing $\frac{d}{dt}_{|t=0}Dir(f^t;\Omega)$ with $\frac{d}{dt}_{|t=0}Dir(\widetilde{f^t};\Omega)$, where $\widetilde{f^t}$ is a smooth perturbation of f (see (3.64), (3.65) and (3.66) for details). Note, f, f^t and $\Lambda_k(x)\cdot(\Gamma_k)_{\#}\left(f(x)\right)$ all belong to the Sobolev space $\mathcal{Y}_2(\Omega,\mathbf{Q}_Q(\mathbb{R}^n))$, i.e., $\boldsymbol{\xi}\circ f$, $\boldsymbol{\xi}\circ f^t$ and $\boldsymbol{\xi}\Big(\Lambda_k(x)\cdot(\Gamma_k)_{\#}\left(f(x)\right)\Big)$ all belong to $W^{1,2}(\Omega)$. By the fine-property of functions in Sobolev spaces (e.g., see [5, 6.1.3]), the approximate derivatives of $\boldsymbol{\xi}\circ f$, $\boldsymbol{\xi}\circ f^t$ and $\boldsymbol{\xi}\Big(\Lambda_k(x)\cdot(\Gamma_k)_{\#}\left(f(x)\right)\Big)$ all exist $\mathcal{L}^2$ a.e. in Ω (e.g., see [5, 6.1.3]). Recall from the description on the `strongly approximately affinely approximable` multiple-valued functions in the Preliminaries of this article (or see [1, 1.4 (3)], if $\boldsymbol{\xi}\circ f$ is approximate differentiable at x and $f_i(x)=f_j(x)$, then

$$\begin{cases} \text{ap } D_x f_i(x) = \text{ap } D_x f_j(x), \\ \text{ap } D_x\left(\Lambda_k(x)\cdot\Gamma_k\circ f_i(x)\right) = \text{ap } D_x\left(\Lambda_k(x)\cdot\Gamma_k\circ f_j(x)\right). \end{cases}$$

Hence, if $\boldsymbol{\xi}\circ f$ is approximate differentiable at x and $f(x)=\sum_{\kappa=1}^{K}\ \ell_\kappa\cdot[\![f_\kappa(x)]\!]$, then

$$\begin{cases} \text{ap } A_x f(x) = \sum_{\kappa=1}^{K}\ \ell_\kappa\cdot[\![\text{ap } D_x f_\kappa(x)]\!], \\ \text{ap } A_x\Big(\Lambda_k(x)\cdot(\Gamma_k)_{\#}\left(f(x)\right)\Big) = \sum_{\kappa=1}^{K}\ \ell_\kappa\cdot[\![\text{ap } D_x\left(\Lambda_k(x)\cdot\Gamma_k\circ f_\kappa(x)\right)]\!]. \end{cases}$$

Therefore, for $\mathcal{L}^2$ a.e. $x\in\Omega$, we have

$$\begin{aligned} &\text{ap } A_x f^t(x) = \sum_{i=1}^{Q}\ [\![\text{ap } D_x f_i^t(x)]\!] \\ &= \sum_{\kappa=1}^{K}\ \ell_\kappa\cdot[\![\text{ap } D_x f_\kappa(x) + t\cdot\text{ap } D_x\left(\Lambda_k(x)\cdot\Gamma_k\circ f_\kappa(x)\right)]\!] \\ &= \sum_{i=1}^{Q}\ [\![\text{ap } D_x f_i(x) + t\cdot\text{ap } D_x\left(\Lambda_k(x)\cdot\Gamma_k\circ f_i(x)\right)]\!]. \end{aligned} \tag{3.62}$$

From [1, Theorem 2.2], we may compute the Dirichlet integral of f^t on an open set Ω by $\sum\limits_{i=1}^{Q} \|\text{ap } D_x f_i^t\|^2_{L^2(\Omega)}$. Thus, from (3.62), we have

$$Dir(f^t;\Omega) = \sum_{i=1}^{Q} \int_\Omega \left|\text{ap } D_x f_i^t(x)\right|^2 \ dx$$

$$= \sum_{i=1}^{Q} \int_\Omega \left|\text{ap } D_x f_i(x) + t\cdot[\text{ap } D_x \Lambda_k(x)\ \Gamma_k(f_i(x)) + \Lambda_k(x)\cdot \text{ap } D_x(\Gamma_k\circ f_i)(x)]\right|^2 \ dx$$

$$= \sum_{i=1}^{Q} \int_\Omega \left|\text{ap } D_x f_i(x)\right|^2 \ dx$$

$$+2t\cdot \sum_{i=1}^{Q} \int_\Omega \langle \text{ap } D_x f_i(x) : \text{ap } D_x\Lambda_k(x)\ \Gamma_k(f_i(x))\rangle \ dx$$

$$+2t\cdot \sum_{i=1}^{Q} \int_\Omega \langle \text{ap } D_x f_i(x) : \Lambda_k(x)\cdot \text{ap } D_y\Gamma_k(f_i(x))\ \text{ap } D_x f_i(x)\rangle \ dx$$

$$+t^2\cdot \sum_{i=1}^{Q} \int_\Omega \left|\text{ap } D_x \Lambda_k(x)\ \Gamma_k(f_i(x)) + \Lambda_k(x)\cdot \text{ap } D_x(\Gamma_k\circ f_i)(x)\right|^2 \ dx.$$

Hence,

$$\frac{d}{dt}_{|t=0} Dir(f^t;\Omega) \tag{3.63}$$

$$= 2\cdot \sum_{i=1}^{Q} \int_\Omega \langle \text{ap } D_x f_i(x) : \text{ap } D_x\Lambda_k(x)\ \Gamma_k(f_i(x))\rangle \ dx$$

$$+2\cdot \sum_{i=1}^{Q} \int_\Omega \langle \text{ap } D_x f_i(x) : \Lambda_k(x)\cdot \text{ap } D_y\Gamma_k(f_i(x))\ \text{ap } D_x f_i(x)\rangle \ dx.$$

By applying the Lusin-type Theorem on approximating functions in Sobolev spaces by C^1-smooth functions (e.g., see [5, Section 6.6 Corollary 2]), for a given $\Lambda_k \in W_0^{1,2}(\Omega,\mathbb{R})$ and any $\delta > 0$, there exists a C^1-smooth function $\widetilde{\Lambda}_k \in C_0^1(\Omega)$ such that

$$\|\widetilde{\Lambda}_k - \Lambda_k\|_{W^{1,2}(\Omega)} \le \delta \tag{3.64}$$

and

$$\mathcal{L}^m\left(\{x : \widetilde{\Lambda}_k(x) \neq \Lambda_k(x) \text{ or } \nabla_x\widetilde{\Lambda}_k(x) \neq \nabla_x\Lambda_k(x)\}\right) \le \delta \tag{3.65}$$

where ∇ denotes the weak differentiation. Note that if $H \in W^{1,p}_{\text{loc}}$, then $\nabla_x H(x) = \text{ap } D_x H(x)$ for $\mathcal{L}^2$ a.e. $x \in \Omega$, from standard theory of Sobolev spaces (e.g., see [5, p.233 Remark (i)]). Let

$$\widetilde{f}^t(x) := \sum_{i=1}^{Q} [\![f_i(x) + t\cdot \widetilde{\Lambda}_k(x)\ \Gamma_k(f_i(x))]\!]. \tag{3.66}$$

Then, by following the same computation as the one in deriving (3.63), we have

(3.67)

$$\begin{aligned}
&\tfrac{d}{dt}_{|t=0} Dir(f^t;\Omega) - \tfrac{d}{dt}_{|t=0} Dir(\widetilde{f^t};\Omega)\\
&= 2\cdot\sum_{i=1}^{Q}\int_\Omega \left\langle \text{ap } D_x f_i(x) : \left(\nabla_x\Lambda_k(x) - \nabla_x\widetilde{\Lambda}_k(x)\right)\ \Gamma_k(f_i(x))\right\rangle\ dx\\
&+2\cdot\sum_{i=1}^{Q}\int_\Omega \left\langle \text{ap } D_x f_i(x) : \left(\Lambda_k(x) - \widetilde{\Lambda}_k(x)\right)\cdot\nabla_y\Gamma_k(f_i(x))\ \text{ap } D_x f_i(x)\right\rangle\ dx.
\end{aligned}$$

As $\delta \to 0^+$, the first term on the R.H.S. of (3.67) tends to zero by applying (3.64), the finiteness of both $\|\Gamma_k\|_{L^\infty}$ and $Dir(f;\Omega)$; meanwhile the second term on the R.H.S. of (3.67) also tends to zero by applying (3.65), the finiteness of $\|\Lambda_k\|_{L^\infty}$, $\|\widetilde{\Lambda}_k\|_{L^\infty}$, $\|\nabla\Gamma_k\|_{L^\infty}$ and the Lebesgue-integrability of $\sum_{i=1}^{Q}|\text{ap } D_x f_i|^2$. Thus, we conclude that

$$\left|\frac{d}{dt}_{|t=0} Dir(f^t;\Omega) - \frac{d}{dt}_{|t=0} Dir(\widetilde{f^t};\Omega)\right| \to 0$$

as $\delta \to 0^+$. Since the vector field $\psi = \widetilde{\Lambda}_k\cdot\Gamma_k \in C_c^\infty(\Omega\times\mathbb{R}^n,\mathbb{R}^n)$ for any $\delta > 0$, from Definition 2 and the assumption of f being weakly harmonic, we have $\frac{d}{dt}_{|t=0} Dir(\widetilde{f^t};\Omega) = 0$. Therefore, we conclude that $\frac{d}{dt}_{|t=0} Dir(f^t;\Omega) = 0$ must hold.

Step 2: Deriving the "local" monotonicity formula.

For simplicity of notations, let

$$G := (F,h) := (\boldsymbol{\xi}_0\circ f, h)$$

and

$$G_k^* := \left(\boldsymbol{\xi}_0(q^{(k)}), h(w^*)\right),\ \forall\ k\in\{0,...,L\}.$$

Notice that, from (3.26), we have

$$d_k^*(x) = |G_k^* - G(x)|,\ \forall\ x\in\Omega_k^*\left(\frac{2}{5}\sigma_k\right). \tag{3.68}$$

Besides, due to the choice of cut-off function λ and the definition of $\Omega_k^*(\rho)$ in (3.50), we have

$$\lambda(\rho - d_k^*(x)) = 0,\ \forall\ x\notin\Omega_k^*(\rho). \tag{3.69}$$

Since both λ and Γ_k in (3.57) are bounded, we have

$$f^t(x)\in\mathbb{B}^{\mathbf{Q}}_{\frac{2}{5}\sigma_k}(q^{(k)})$$

for all $x\in\Omega$, $\rho\le\frac{2}{5}\sigma_k$ and sufficiently small $|t|$. Hence, under this situation, there is a unique way of paring $f^t(x)$ with $q^{(k)}$ in the sense of (3.24). By applying (3.25), we obtain the expression

$$\boldsymbol{\xi}_0\circ f^t(x) = (1 - t\cdot\lambda(\rho - d_k^*(x)))\cdot\boldsymbol{\xi}_0\circ f(x) + t\cdot\lambda(\rho - d_k^*(x)))\cdot\boldsymbol{\xi}_0\left(q^{(k)}\right).$$

In other words,

$$F^t(x) := \boldsymbol{\xi}_0 \circ f^t(x) = F(x) + t \cdot \lambda(\rho - d_k^*(x)) \cdot \left(\boldsymbol{\xi}_0(q^{(k)}) - F(x)\right)$$

for all $x \in \Omega$, $\rho \leq \frac{2}{5}\sigma_k$ and sufficiently small $|t|$. Furthermore, by letting

$$h^t(x) := h(x) + t \cdot \lambda(\rho - d_k^*(x)) \cdot (h(w^*) - h(x))$$

we obtain the perturbation formula of G,

$$G^t(x) := (F^t(x), h^t(x)) = G(x) + t \cdot \lambda(\rho - d_k^*(x)) \cdot (G_k^* - G(x)) \,. \tag{3.70}$$

Note that in the rest of this paper, the perturbation formula of f and G will be applied only as $\rho \in (\rho_k, \frac{2}{5}\sigma_k)$ for $k \in \{0, 1, ..., k_0 - 1\}$ and $\rho \in (\rho_{k_0}, \frac{2}{5}\min\{\tau_*, \sigma_{k_0}\})$. Due to (3.69) and (3.53), the perturbations in (3.70) leave the boundary value of G fixed, i.e.,

$$G^t(x) = G(x), \ \forall \ x \in \partial\mathbb{U}_r(w).$$

Since f is a stationary-harmonic multiple-valued function and h is a harmonic (single-valued) function,

$$\frac{d}{dt}_{\lfloor t=0} Dir(F^t; \Omega) = 0 = \frac{d}{dt}_{\lfloor t=0} Dir(h^t; \Omega).$$

Thus,

$$\frac{d}{dt}_{\lfloor t=0} Dir(G^t; \Omega) = \frac{d}{dt}_{\lfloor t=0} Dir(f^t; \Omega) + \frac{d}{dt}_{\lfloor t=0} Dir(h^t; \Omega) = 0.$$

From (3.68), this implies that

$$\begin{aligned} 0 &= -\tfrac{1}{2} \tfrac{d}{dt}_{\lfloor t=0} Dir(G^t; \Omega) \\ &= \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho)} \lambda\left(\rho - |G_k^* - G(x)|\right) \cdot |\nabla G(x)|^2 \ dx \\ &- \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho)} \tfrac{\lambda'(\rho - |G_k^* - G(x)|)}{|G_k^* - G(x)|} \cdot \langle \nabla G(x), G_k^* - G(x) \rangle^2 \ dx. \end{aligned}$$

Hence,

$$\begin{aligned} &\int\limits_{\mathbb{U}_r(w)} \lambda\left(\rho - d_k^*(x)\right) \cdot |\nabla G(x)|^2 \ dx \\ &= \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho)} \lambda\left(\rho - |G_k^* - G(x)|\right) \cdot |\nabla G(x)|^2 \ dx \\ &= \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho)} \tfrac{\lambda'(\rho - |G_k^* - G(x)|)}{|G_k^* - G(x)|} \cdot \langle \nabla G(x), G_k^* - G(x) \rangle^2 \ dx \\ &\leq \tfrac{\rho}{2} \cdot \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho)} \lambda'\left(\rho - |G_k^* - G(x)|\right) \cdot |\nabla G(x)|^2 \ dx \\ &= \tfrac{\rho}{2} \cdot \int\limits_{\mathbb{U}_r(w)} \lambda'\left(\rho - d_k^*(x)\right) \cdot |\nabla G(x)|^2 \ dx \end{aligned} \tag{3.71}$$

where the inequality comes from applying the weak conformality conditions of G, which is proved in Proposition 1. To be more precise, when $a, b : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^N$ are two differentiable vector-valued functions, we first note that

$$\sum_{i=1}^{2} \left\langle \partial_i a, \frac{b}{|b|} \right\rangle^2 = \sum_{i=1}^{2} \left\langle \partial_i a, \mathbb{P}_{\{\partial_1 a, \partial_2 a\}} \left(\frac{b}{|b|}\right) \right\rangle^2 \leq \sum_{i=1}^{2} \langle \partial_i a, T_b \rangle^2$$

where $\mathbb{P}_{\{\partial_1 a, \partial_2 a\}}(v)$ denotes the orthogonal projection of vector v into the two-dimensional plane spanned by $\partial_1 a$ and $\partial_2 a$ and $T_b := \frac{\mathbb{P}_{\{\partial_1 a, \partial_2 a\}}\left(\frac{b}{|b|}\right)}{\left|\mathbb{P}_{\{\partial_1 a, \partial_2 a\}}\left(\frac{b}{|b|}\right)\right|}$. From the conformality, i.e., $|\partial_1 a| = |\partial_2 a|$ and $\langle \partial_1 a, \partial_2 a\rangle = 0$, we have $T_b = \frac{\partial_1 a}{|\partial_1 a|} \cdot \cos\theta + \frac{\partial_2 a}{|\partial_2 a|} \cdot \sin\theta$ for some θ. Thus,

$$\sum_{i=1}^{2} \langle \partial_i a, T_b\rangle^2 = |\partial_1 a|^2 = \frac{1}{2}\left(|\partial_1 a + \partial_2 a|^2\right) = \frac{1}{2}|\nabla a|^2.$$

Therefore, we conclude that

$$\sum_{i=1}^{2} \langle \partial_i a, b\rangle^2 = |b|^2 \sum_{i=1}^{2} \left\langle \partial_i a, \frac{b}{|b|}\right\rangle^2 \leq \frac{1}{2}|\nabla a|^2 |b|^2.$$

Let

$$\Psi_k(\rho) := \int\limits_{\mathbb{U}_r(w)} \lambda\left(\rho - d_k^*(x)\right) \cdot |\nabla G(x)|^2 \, dx.$$

Then (3.71) gives

$$\Psi_k(\rho) \leq \frac{\rho}{2}\frac{d}{d\rho}\Psi_k(\rho)$$

for all $\rho \in (\rho_k, \frac{2}{5}\sigma_k)$ for $k \in \{0, 1, ..., k_0 - 1\}$ or $\rho \in (\rho_{k_0}, \frac{2}{5}\min\{\tau_*, \sigma_{k_0}\})$. This inequality implies the nondecreasing property of $\frac{\Psi_k(\rho)}{\rho^2}$, i.e., if $s \leq t$, then

$$\frac{\Psi_k(s)}{s^2} \leq \frac{\Psi_k(t)}{t^2} \tag{3.72}$$

where either $s, t \in (\rho_k, \frac{2}{5}\sigma_k)$ as $k \in \{0, ..., k_0 - 1\}$ or $s, t \in (\rho_{k_0}, \frac{2}{5}\min\{\tau_*, \sigma_{k_0}\})$ as $k = k_0$.

Step 3: Extending the monotonicity formula "globally".

The estimates of τ_* by the Dirichlet integral of f rely on applying the monotonicity formula (3.72) on each admissible closed ball $\mathbb{B}_s^{\mathbf{Q}}(q^{(k)})$, for each fixed $k \in \{0, ..., k_0\}$, and keeping the nested condition,

$$\mathbb{B}_s^{\mathbf{Q}}(q^{(k)}) \subset \mathbb{B}_t^{\mathbf{Q}}(q^{(k+1)}) \tag{3.73}$$

by proper choices of s and t,

$$\left(\rho_k, \frac{2}{5}\sigma_k\right) \ni s < t \in \left(\rho_{k+1}, \frac{2}{5}\min\{\tau_*, \sigma_{k+1}\}\right) \tag{3.74}$$

for all $k \in \{0, ..., k_0 - 1\}$. However, one can't apply the monotonicity formula of $\frac{\Psi_k(\rho)}{\rho^2}$ on the whole interval of $\left(\rho_k, \frac{2}{5}\min\{\tau_*, \sigma_k\}\right)$. This is because that the parameter ρ in $\lambda(\rho - d_k^*(\cdot))$ doesn't exactly represent the radius of an admissible closed ball $\mathbb{B}_\rho^{\mathbf{Q}}(\cdot)$ (note that there is an "error" term, the harmonic function h, in the definition of $d_k^*(\cdot)$). Moreover, one needs to change the center of admissible closed balls, i.e., from $q^{(k)}$ to $q^{(k+1)}$, and keep the nested condition (3.73) to establish a relation (which is nearly an inequality) between $\frac{\Psi_k(s)}{s^2}$ and $\frac{\Psi_k(t)}{t^2}$ for s and t fulfilling (3.74). But the parameter ρ in $\lambda(\rho - d_k^*(\cdot))$ only (nearly) represents the

distance between $f(x)$ and $q^{(k)}$ instead of $f(x)$ and $q^{(0)}$. Hence, in order to keep the condition, $\Omega_k^*(s) \subset\subset \mathbb{U}_r(w)$, one needs to choose the upper bound of s in (3.74).

Let λ be the cut-off function defined in (3.56) and ε in (3.56) is any number satisfying

$$0 < \varepsilon < \frac{\min\{\sigma_0, \tau_*\}}{10}. \tag{3.75}$$

Recall, from (3.29), that $10Q \cdot \rho_k \le \sigma_k$ for each fixed $k \in \{0, ..., L\}$ and, from (3.48), that $k_0 \in \{0, ..., L\}$ is the unique one fulfilling $10Q \cdot \rho_{k_0} \lneqq \tau_* \le 10Q \cdot \rho_{k_0+1}$. Let ρ satisfy $\varepsilon < \rho < \frac{2}{5}\min\{\sigma_0, \tau_*\}$. Note that, from (3.53), $\Omega_0^*(\rho) \subset\subset \mathbb{U}_r(w)$, if ρ fulfills the condition $\rho < \min\{\sigma_0, \tau_*\}$. Observe that

(3.76)
$$\rho^{-2} \cdot \Psi_0(\rho) = \rho^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_0^*(\rho)} \lambda\left(\rho - d_0^*(x)\right) \cdot \left|\nabla G(x)\right|^2 \, dx$$

$$\ge \rho^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_0^*(\rho-\varepsilon)} \lambda\left(\rho - d_0^*(x)\right) \cdot \left|\nabla G(x)\right|^2 \, dx = \rho^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_0^*(\rho-\varepsilon)} \left|\nabla G(x)\right|^2 \, dx.$$

By applying the monotonicity formula (3.72) to the L.H.S. of (3.76) and letting $\varepsilon \to 0$ on the R.H.S. of (3.76), we derive

$$\rho^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_0^*(\rho)} \left|\nabla G(x)\right|^2 \, dx \le t^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_0^*(t)} \left|\nabla G(x)\right|^2 \, dx \tag{3.77}$$

for $0 < \rho < t < \frac{2}{5}\min\{\sigma_0, \tau_*\}$. Now, since $w^* \in A$ is also a "good" point of G (see Remark 2), we may apply Lemma 1 to the L.H.S. of (3.77) and let $t \to \frac{2}{5}\min\{\sigma_0, \tau_*\}$ on the R.H.S. of (3.77) to derive

$$2\pi \le \left(\frac{2}{5}\min\{\sigma_0, \tau_*\}\right)^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_0^*(\frac{2}{5}\min\{\sigma_0,\tau_*\})} \left|\nabla G(x)\right|^2 \, dx. \tag{3.78}$$

Case 1° $k_0 = 0$**.**

As $\tau_* \le \sigma_0$, we derive from (3.78),

$$\tau_* \le \frac{5}{2} \cdot \sqrt{\frac{Dir(G; \mathbb{U}_r(w))}{2\pi}}. \tag{3.79}$$

As $\sigma_0 < \tau_* \le 10Q \cdot \rho_1$, we apply (3.31) in Proposition 3 (i.e., $\rho_1 < C_0(n, Q) \cdot \sigma_0$), and (3.78) to derive

$$\tau_* < 25Q \cdot C_0(n, Q) \cdot \sqrt{\frac{Dir(G; \mathbb{U}_r(w))}{2\pi}}. \tag{3.80}$$

Thus, from (3.79) and (3.80), we conclude that in this case,

$$\mathcal{G}\left(f(w^*), f_{|\partial \mathbb{U}_r(w)}\right) < \left(\frac{5}{2} + 25Q \cdot C_0(n, Q)\right) \cdot \sqrt{\frac{Dir(G; \mathbb{U}_r(w))}{2\pi}}. \tag{3.81}$$

Case 2° $k_0 \ge 1$**.**

We also need to adjust the range of the parameter ρ in the definition of distance function $d_k^*(\cdot)$ to take care of the "error" term coming from the harmonic function

h in $d_k^*(\cdot)$. Recall from (3.32) the conditions of nested admissible closed balls that, for any $k \in \{1, ..., L\}$,

$$\mathbb{B}_s^{\mathbf{Q}}(q^{(k-1)}) \subset \mathbb{B}_t^{\mathbf{Q}}(q^{(k)}), \text{ if } s \leq \sigma_{k-1} \text{ and } t \geq \rho_k. \tag{3.82}$$

Hence, from (3.82), $\mathcal{G}(q^{(k)}, f(x)) \leq \rho_k$, if $x \in \Omega_{k-1}^*(\sigma_{k-1})$. Recall from the definition of $\Omega_k^*(\cdot)$ that if $x \in \Omega_{k-1}^*(\sigma_{k-1})$, then $\mathcal{G}(q^{(k-1)}, f(x)) \leq \sigma_{k-1}$ and $|h(w^*) - h(x)| \leq \sigma_{k-1}$. Thus, we have

$$d_k^*(x) \leq \mathcal{G}(q^{(k)}, f(x)) + |h(w^*) - h(x)| \leq \rho_k + \sigma_{k-1}, \ \forall\ x \in \Omega_{k-1}^*(\sigma_{k-1}).$$

In other words,

$$s \leq \sigma_{k-1} < \rho_k + \sigma_{k-1} \leq t \ \Rightarrow\ \Omega_{k-1}^*(s) \subset \Omega_k^*(t) \tag{3.83}$$

for all $k \in \{1, ..., k_0\}$.

From the definition of k_0 and the assumption $k_0 \geq 1$, we have $\sigma_0 < \rho_1 < 10Q \cdot \rho_1 \lneqq \tau_*$. Hence, from (3.75), we have $\varepsilon \in (0, \frac{\sigma_0}{10})$. Note, from (3.53), we have

$$\Omega_k^*\left(\frac{2}{5}\min\{\tau_*, \sigma_k\}\right) \subset\subset \mathbb{U}_r(w), \ \forall\ k \in \{0, ..., k_0\}.$$

From Proposition 3 on the sequences of $\{\rho_k\}_{k=0}^L$ and $\{\sigma_k\}_{k=0}^L$, we have the inequality,

$$\rho_k + \sigma_{k-1} + \frac{\sigma_0}{10} < 3Q\rho_k < 4Q\rho_k \leq \frac{2}{5}\min\{\tau_*, \sigma_k\} \tag{3.84}$$

for all $k \in \{1, ..., k_0\}$. Hence for all $k \in \{1, ..., k_0\}$, if $\varepsilon \in (0, \frac{\sigma_0}{10})$, then $\rho_k + \sigma_{k-1} + \varepsilon < \frac{2}{5}\min\{\tau_*, \sigma_k\}$.

Now for each $k \in \{1, ..., k_0\}$, we consider $\rho \in (\rho_k + \sigma_{k-1} + \varepsilon, \frac{2}{5}\min\{\tau_*, \sigma_k\})$ below. From

$$\begin{aligned} &\rho^{-2} \cdot \Psi_k(\rho) = \rho^{-2} \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho)} \lambda\left(\rho - d_k^*(x)\right) \cdot |\nabla G(x)|^2 \ dx \\ &\geq \rho^{-2} \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho - \varepsilon)} \lambda\left(\rho - d_k^*(x)\right) \cdot |\nabla G(x)|^2 \ dx = \rho^{-2} \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho - \varepsilon)} |\nabla G(x)|^2 \ dx \end{aligned} \tag{3.85}$$

we again apply the monotonicity formula (3.72) to the L.H.S. of (3.85) and letting $\varepsilon \to 0$ on the R.H.S. of (3.85) to derive

$$\rho^{-2} \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(\rho)} |\nabla G(x)|^2 \ dx \leq t^{-2} \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(t)} |\nabla G(x)|^2 \ dx \tag{3.86}$$

for $\rho_k + \sigma_{k-1} + \frac{\sigma_0}{10} < \rho < t < \frac{2}{5}\min\{\tau_*, \sigma_k\}$. Now we let let $\rho = 3\rho_k$ on the L.H.S. of (3.86) and $t \to \frac{2}{5}\min\{\tau_*, \sigma_k\}$ on the R.H.S. of (3.86), we have

$$\begin{aligned} &(3\rho_k)^{-2} \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*(3\rho_k)} |\nabla G(x)|^2 \ dx \\ &\leq \left(\tfrac{2}{5}\min\{\tau_*, \sigma_k\}\right)^{-2} \int\limits_{\mathbb{U}_r(w) \cap \Omega_k^*\left(\frac{2}{5}\min\{\tau_*, \sigma_k\}\right)} |\nabla G(x)|^2 \ dx. \end{aligned} \tag{3.87}$$

Hence, for each $k \in \{1, ..., k_0\}$, we have

$$
\begin{aligned}
&\left(\tfrac{2}{5}\min\{\tau_*,\sigma_k\}\right)^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_k^*(\frac{2}{5}\min\{\tau_*,\sigma_k\})} |\nabla G(x)|^2 \ dx \\
&\geq (3\rho_k)^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_k^*(3\rho_k)} |\nabla G(x)|^2 \ dx \\
&\geq (3C_0(n,Q)\cdot\sigma_{k-1})^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_{k-1}^*(\frac{2}{5}\sigma_{k-1})} |\nabla G(x)|^2 \ dx
\end{aligned} \tag{3.88}
$$

where the first inequality comes from applying (3.87) and the second inequality comes from applying (3.83), (3.31). Then, by applying (3.88) inductively on $k = k_0, ..., 1$, we have

$$
\begin{aligned}
&\left(\tfrac{2}{5}\min\{\tau_*,\sigma_{k_0}\}\right)^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_{k_0}^*(\frac{2}{5}\min\{\tau_*,\sigma_{k_0}\})} |\nabla G(x)|^2 \ dx \\
&\geq \left(\tfrac{2}{15C_0(n,Q)}\right)^{2k_0-2}\cdot\left(\tfrac{2}{5}\sigma_0\right)^{-2} \int\limits_{\mathbb{U}_r(w)\cap\Omega_0^*(\frac{2}{5}\sigma_0)} |\nabla G(x)|^2 \ dx.
\end{aligned} \tag{3.89}
$$

Thus, by applying (3.78) to the R.H.S. of (3.89), we have

$$
\min\{\tau_*,\sigma_{k_0}\} \leq \frac{5}{2}\left(\frac{15C_0(n,Q)}{2}\right)^{k_0-1}\cdot\sqrt{\frac{Dir\,(G;\mathbb{U}_r(w))}{2\pi}}. \tag{3.90}
$$

As $\tau_* \leq \sigma_{k_0}$, we derive from (3.90) that

$$
\tau_* \leq \frac{5}{2}\left(\frac{15C_0(n,Q)}{2}\right)^{k_0-1}\cdot\sqrt{\frac{Dir\,(G;\mathbb{U}_r(w))}{2\pi}}. \tag{3.91}
$$

As $\tau_* > \sigma_{k_0}$, (3.90) gives

$$
\sigma_{k_0} \leq \frac{5}{2}\left(\frac{15C_0(n,Q)}{2}\right)^{k_0-1}\cdot\sqrt{\frac{Dir\,(G;\mathbb{U}_r(w))}{2\pi}}. \tag{3.92}
$$

Since the choice of k_0 implies $\tau_* \leq 10Q\cdot\rho_{k_0+1}$, from (3.31) in Proposition 3 (i.e., $\rho_{k+1} < C_0(n,Q)\cdot\sigma_k$) and (3.92), we have

$$
\tau_* \leq 25Q\cdot C_0(n,Q)\cdot\left(\frac{15C_0(n,Q)}{2}\right)^{k_0-1}\cdot\sqrt{\frac{Dir\,(G;\mathbb{U}_r(w))}{2\pi}}. \tag{3.93}
$$

Since $C_0(n,Q) > 1$ and $k_0 \leq Q-1$ (see Proposition 3), from (3.81), (3.91) and (3.93), we conclude that

$$
\mathcal{G}\left(f(w^*), f_{|\partial\mathbb{U}_r(w)}\right) =: \tau_* \leq \sqrt{\frac{Dir(G;\mathbb{U}_r(w))}{2\pi\cdot\delta(n,Q)}} \tag{3.94}
$$

for some constant $\delta(n,Q) > 0$.

□

3.3. **Proof of Theorem 1.** To prove interior continuity of f, we may assume without loss of generality that $\Omega = \mathbb{U}_{R_0}(0) \subset \mathbb{R}^2$, an open ball of radius $R_0 > 0$ with center at the origin of $\mathbb{R}^2$. Since $f \in \mathcal{Y}_2(\mathbb{U}_{R_0}(0), \mathbf{Q}_Q(\mathbb{R}^n))$ means $\boldsymbol{\xi} \circ f \in W^{1,2}(\mathbb{U}_{R_0}(0), \mathbb{R}^N)$, by Courant-Lebesgue Lemma (see Lemma 2), for any $\mathbb{U}_R(w) \subset\subset \mathbb{U}_{R_0}(0)$, one may choose a proper slice of f by $\partial \mathbb{U}_r(w)$ for some $r \in \left[\frac{R}{2}, R\right]$ such that $\boldsymbol{\xi} \circ f$ is continuous on the compact set $\partial \mathbb{U}_r(w)$ and the oscillation of $\boldsymbol{\xi} \circ f$ is bounded by $C(n, Q) \cdot \sqrt{Dir\,(\boldsymbol{\xi} \circ f; \mathbb{U}_R(w))}$. Thus, by the bi-Lipschitz continuity of $\boldsymbol{\xi}$, we conclude that the multiple-valued function f is uniformly continuous on $\partial \mathbb{U}_r(w)$ and

$$\operatorname*{osc}_{\partial \mathbb{U}_r(w)} f \leq C(n, Q, \mathrm{Lip}(\boldsymbol{\xi}), \mathrm{Lip}(\boldsymbol{\xi}^{-1})) \cdot \sqrt{Dir\,(f; \mathbb{U}_R(w))} =: \alpha_1(R).$$

On the other hand, for any $w^* \in A \cap \mathbb{U}_r(w)$, where A is a set of Lebesgue points of $|\nabla(\boldsymbol{\xi}_0 \circ f)|^2$, we apply Lemma 3 to derive

(3.95)
$$\inf_{x \in \partial \mathbb{U}_r(w)} \mathcal{G}(f(x), f(w^*)) \leq \sqrt{\frac{Dir\,(G; \mathbb{U}_r(w))}{2\pi \cdot \delta(n, Q)}} \leq \frac{\sqrt{Dir\,(f; \mathbb{U}_R(w))} + \|\nabla h\|_{L^2(\mathbb{U}_R(w))}}{\sqrt{2\pi \cdot \delta(n, Q)}}.$$

Note that $\|\nabla h\|_{L^2(\mathbb{U}_R(w))}$ in (3.95) depends on the choice of $\boldsymbol{\xi}_0$ because one needs to choose a proper coordinates of $\mathbb{R}^n$ for a given w^*. Hence, we need to have a control of $\|\nabla h\|_{L^2(\mathbb{U}_R(w))}$ in (3.95) when h is induced from a distinct Lipschitz correspondence $\boldsymbol{\xi}_0$.

Let $\overset{\star}{\boldsymbol{\xi}}_0$ be a *fixed* Lipschitz correspondence and $\overset{\star}{h} : \mathbb{U}_{R_0}(0) \to \mathbb{R}^2$ be the harmonic function induced from the Hopf differential of $\overset{\star}{\boldsymbol{\xi}}_0 \circ f$. Suppose $h : \mathbb{U}_{R_0}(0) \to \mathbb{R}^2$ is a harmonic function with respective to an arbitrarily chosen Lipschitz correspondence $\boldsymbol{\xi}_0$. We would like to estimate the oscillation of $\|\nabla h\|_{L^2(\mathbb{U}_R(w))} - \|\nabla \overset{\star}{h}\|_{L^2(\mathbb{U}_R(w))}$ below. By a simple computation from (3.7), we have

$$|\nabla h|^2 = \frac{|\varphi|^2}{8} + 2.$$

From applying (3.4) and (3.9), we have

$$||\nabla h|^2 - |\nabla \overset{\star}{h}|^2| \leq \tfrac{1}{8} \cdot |\varphi - \overset{\star}{\varphi}| \cdot |\varphi + \overset{\star}{\varphi}| \leq \tfrac{1}{8} \cdot |\varphi - \overset{\star}{\varphi}| \cdot (|\overset{\star}{\varphi} - \varphi| + 2|\overset{\star}{\varphi}|)$$

$$\leq 2 \cdot C_{R_0}^2 + 2 \cdot C_{R_0} \cdot |\nabla(\overset{\star}{\boldsymbol{\xi}}_0 \circ f)|^2$$

where $C_{R_0} := \frac{Dir(f; \mathbb{U}_{R_0}(0))}{\pi R_0^2}$. Thus,

$$\begin{aligned}\int_{\mathbb{U}_R(w)} |\nabla h|^2 \, dx &\leq \int_{\mathbb{U}_R(w)} |\nabla \overset{\star}{h}|^2 \, dx + \int_{\mathbb{U}_R(w)} ||\nabla h|^2 - |\nabla \overset{\star}{h}|^2| \, dx \\ &\leq Dir(\overset{\star}{h}; \mathbb{U}_R(w)) + 2\pi C_{R_0}^2 R^2 + 2C_{R_0} \cdot Dir(f; \mathbb{U}_R(w)) =: \beta(R),\end{aligned} \tag{3.96}$$

where $\beta(R) \to 0$ as $R \to 0$. Notice that $\beta(R)$ defined on the R. H. S. of (3.96) is independent of the choice of Lipschitz correspondence $\boldsymbol{\xi}_0$, although $\int_{\mathbb{U}_R(w)} |\nabla h|^2 \, dx$ on the L. H. S. of (3.96) does depend on $\boldsymbol{\xi}_0$. Hence, from (3.95), we obtain

$$\inf_{x \in \partial \mathbb{U}_r(w)} \mathcal{G}(f(x), f(w^*)) \leq \frac{\sqrt{Dir\,(f; \mathbb{U}_R(w))} + \sqrt{\beta(R)}}{\sqrt{2\pi \cdot \delta(n, Q)}} =: \alpha_2(R)$$

for any $w^* \in \mathbb{U}_r(w) \cap A$. Therefore,

$$\operatorname*{osc}_{\mathbb{U}_r(w) \cap A} f \leq 4 \max\{\alpha_1(R), \alpha_2(R)\}.$$

Note that $\mathcal{L}^2(\mathbb{U}_r(w) \setminus A) = 0$ and $\alpha_1(R)$ and $\alpha_2(R)$ tend to zero as $R \to 0$. This proves the continuity of f at w, and the proof of Theorem 1 is finished.

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